DUALITY METHODS AND PERTURBATION OF SEMIGROUPS

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1. **Introduction.** In [5], the author announced several theorems applying the semi-inner product methods of Lumer and Phillips [4] to the perturbation theory of one-parameter holomorphic contraction semigroups on Banach spaces. This note extends the methods to a perturbation theorem of Trotter [9], with proofs, and announces generalizations to locally convex spaces. (See also Kato [3].)

2. **Generation theorem, ϕ-sectorial operators.** Let $\mathfrak{X}$ be a complex Banach space, and let $[\ ,\ ]: \mathfrak{X}\times\mathfrak{X}\rightarrow \mathbf{C}$ be a semi-inner product for $\mathfrak{X}$, in the sense of [4]: (i) for all $v\in\mathfrak{X}$, $u\rightarrow[u, v]$ is a linear functional on $\mathfrak{X}$, (ii) $[u, u]\geqq 0$ for all $u\in\mathfrak{X}$, with $\|u\|=[u, u]^{1/2}$, and (iii) $|[u, v]| <\|u\|\,\|v\|$.

DEFINITION 1. A linear operator A with domain $\mathfrak{D}(A)\subset\mathfrak{X}$ is ϕ-*sectorial* for $0\leqq\phi\leqq\pi/2$ iff for every $u\in\mathfrak{D}(A)$,

$$\tan\phi\,|\operatorname{Im}[Au, u]| \leqq -\operatorname{Re}[Au, u] \geqq 0. \tag{1}$$

Every ϕ_1-sectorial operator is ϕ_2-sectorial for all $\phi_2\leqq\phi_1$, and every 0-sectorial operator is dissipative (Re $[Au, u]\leqq 0$ as in [4]). If $\phi=\pi/2$, replace the first inequality by Im $[Au, u]=0$. If $\Delta_\phi=\{z \mid \pi\geqq|\arg z|\geqq\pi/2+\phi\}$, and $W(A)=\{[Au, u] \mid u\in\mathfrak{D}(A), \|u\|=1\}$ is the numerical range of A then A is ϕ-sectorial iff $\Delta_\phi\supset\{W(A)\}^-$ (obvious when sketched).

DEFINITION 2. A one-parameter semigroup T is in the family $CH(\phi)$ of *holomorphic contraction semigroups* on the sector $S_\phi =\{z \mid |\arg z|\leqq\phi\}$ iff

(a) T is a homomorphism of the additive semigroup of S_ϕ into the multiplicative semigroup $\mathfrak{C}(\mathfrak{X})$ of all contraction operators on $\mathfrak{X}$ ($\|T(z)\|\leqq 1$),

(b) $z\rightarrow T(z)$ is a holomorphic function from $\operatorname{int}(S_\phi)$ to $\mathfrak{C}(\mathfrak{X})\subset\mathfrak{L}(\mathfrak{X})$, the Banach algebra of bounded operators on $\mathfrak{X}$ (see [2, Chapter 5]), and

(c) (slightly redundant) for all $u\in\mathfrak{X}$, the map $z\rightarrow T(z)u$ is continuous from S_ϕ into $\mathfrak{X}$.

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