THE UNIQUENESS OF THE (COMPLETE) NORM TOPOLOGY

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Communicated by Richard Arens, March 7, 1967

In this paper we show that every semisimple Banach algebra over $\boldsymbol{R}$ or $\boldsymbol{C}$ has the uniqueness of norm property, that is we show that if $\mathfrak{A}$ is a Banach algebra with each of the norms $\|\ \|$, $\|\ \|'$ then these norms define the same topology. This result is deduced from a maximum property of the norm in a primitive Banach algebra (Theorem 1).

In the following $\boldsymbol{F}$ is a field which may be taken throughout as $\boldsymbol{R}$, the real field, or $\boldsymbol{C}$, the complex field. If $\mathfrak{X}$ is a normed space then $\mathfrak{B}(\mathfrak{X})$ will denote the space of bounded linear operators on $\mathfrak{X}$.

LEMMA 1. *Let F, G be closed subspaces of the Banach space E such that $F+G=E$. Then there exists $L>0$ such that if $x\in E$ then there is an $f\in F$ with*

(i) $\|f\| \leqq L\|x\|$.

(ii) $x-f\in G$.

PROOF. The map $(f, g)\to f+g$ is a continuous map of $F\oplus G$ onto E and so is open by the open mapping theorem [1, p. 34]. Thus there is $\delta>0$ such that if $y\in E$ with $\|y\|<\delta$ then there are f', $g'\in G$ with $\|f'\|$, $\|g'\|\leqq 1$ and $f'+g'=y$. The result of the lemma then follows if we take $L=\delta^{-1}$, $y=x\|x\|^{-1}\delta$ and $f=f'L\|x\|$.

THEOREM 1. *Let $\mathfrak{A}$ be a Banach algebra over $\boldsymbol{F}$ and let $\mathfrak{X}$ be a normed space over $\boldsymbol{F}$. Suppose that $\mathfrak{X}$ is a faithful strictly irreducible left $\mathfrak{A}$-module and that the maps $\xi\to a\xi$ from $\mathfrak{X}$ into $\mathfrak{X}$ are continuous for each $a\in\mathfrak{A}$. Then there exists a constant M such that*

$$\|a\xi\|' \leqq M\|a\|\cdot\|\xi\|'$$

for all $a\in\mathfrak{A}$, $\xi\in\mathfrak{X}$, where $\|\cdot\|$ is the norm in $\mathfrak{A}$ and $\|\cdot\|'$ the norm in $\mathfrak{X}$.

The theorem asserts that the natural map $\mathfrak{A}\to\mathfrak{B}(\mathfrak{X})$ is continuous. It is a much stronger version of [4, Theorem 2.2.7] but applicable only to primitive algebras. It would be interesting to know how far it can be generalized.

PROOF. If $\xi\in\mathfrak{X}$ and $a\to a\xi(\mathfrak{A}\to\mathfrak{X})$ is continuous then the map $a\to ab\to ab\xi$, being a composition of continuous maps, is continuous. Since $\mathfrak{X}$ is strictly irreducible, if $\xi\neq 0$ we can, by a suitable choice of b, make $b\xi$ any particular vector in $\mathfrak{X}$ and so if $a\to a\xi$ is continuous for one nonzero ξ it is continuous for all ξ in $\mathfrak{X}$. We shall deduce a contradic-