# SEMI-IDEMPOTENT MEASURES ON ABELIAN GROUPS[1]

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Communicated by E. Hewitt, September 26, 1966

Let $M(G)$ denote the set of complex valued regular Borel measures on a compact abelian group $G$. We assume that $\Gamma$, the dual group of $G$, is a totally ordered group. Let $F(G)$ denote the set of all $\mu \in M(G)$ such that the Fourier transform $\hat{\mu}$ of $\mu$ is an integer valued function. A measure $\mu$ is idempotent if $\hat{\mu}$ assumes only the values 0 or 1. A $\mu \in M(G)$ is semi-idempotent if $\hat{\mu}(\gamma) = 0$ or 1 for all $\gamma > 0$ in $\Gamma$.

The purpose of this note is to sketch a proof of the following theorem.

THEOREM 1. *If $\mu \in M(G)$ and $\hat{\mu}(\gamma)$ is an integer for all $\gamma > 0$ in $\Gamma$, then there exists a $\lambda \in F(G)$ such that $\hat{\lambda}(\gamma) = \hat{\mu}(\gamma)$ for all $\gamma > 0$. In particular, if $\mu$ is a semi-idempotent measure on $G$, then there exists an idempotent measure $\lambda$ on $G$ such that $\hat{\mu}(\gamma) = \hat{\lambda}(\gamma)$ for all $\gamma > 0$ in $\Gamma$.*

This result was obtained by Helson [**3**], for the case $G = T$, the circle group, $\Gamma = Z$ the integer group. Also a special case of Theorem 1 for $G = T^2$, $\Gamma = Z^2$ was proven by Rudin [**6**].

OUTLINE OF PROOF. We assume first that $G = T^k$, the $k$-dimensional torus group and $\Gamma = Z^k$ is a totally ordered group. Then $Z^k = \Gamma_1 \oplus \Gamma_2$ where $\Gamma_2$ is a subgroup of the reals, $\Gamma_1$ is a totally ordered group, and $\Gamma_1 \oplus \Gamma_2$ is lexicographically ordered from the right. Let $\mu \in M(G)$ such that $\hat{\mu}(\gamma)$ is an integer for all $\gamma > 0$ in $Z^k$.

Let $E(\mu) = \{\gamma \in Z^k : \gamma > 0 \text{ and } \hat{\mu}(\gamma) \neq 0\}$ and, for every positive integer $n$, let

$$A_n = \{\gamma \cdot \mu \in M(G) : \gamma = (\gamma_1, \gamma_2) \in E(\mu) \text{ and } \gamma_2 > n\}.$$

We then prove the following

LEMMA. *If $E(\mu)$ and $A_n$ are defined as above, we have either*

*(1) $E(\mu)$ is contained in a finite union of $(k-1)$-dimensional hyperplanes, or*

*(2) $A_n \neq \emptyset$ for every $n$.*

If (1), then Theorem 1 follows by induction. If (2), we set

[1] This is an announcement of a portion of the author's dissertation at the University of Wisconsin written under the direction of Professor Walter Rudin.