TWO-SIDED IDEALS IN C^*-ALGEBRAS

BY ERLING STØRMER

Communicated by R. Arens, October 28, 1966

If $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{I}$ and $\mathfrak{F}$ are uniformly closed two-sided ideals in $\mathfrak{A}$ then so is $\mathfrak{I}+\mathfrak{F}$. The following problem has been proposed by J. Dixmier [1, Problem 1.9.12]: is $(\mathfrak{I}+\mathfrak{F})^+=\mathfrak{I}^++\mathfrak{F}^+$, where $\mathfrak{K}^+$ denotes the set of positive operators in a family $\mathfrak{K}$ of operators? He suggested to the author that techniques using the duality between invariant faces of the state space $S(\mathfrak{A})$ of $\mathfrak{A}$ and two-sided ideals in $\mathfrak{A}$, as shown by E. Effros, might be helpful in studying it. In this note we shall use such arguments to solve the problem to the affirmative.

By a *face* of $S(\mathfrak{A})$ we shall mean a convex subset F such that if $\rho\in F$, $\omega\in S(\mathfrak{A})$ and $a\omega\leqq\rho$ for some $a>0$, then $\omega\in F$. F is an *invariant face* if $\rho\in F$ implies the state $B\rightarrow\rho(A^*BA)\cdot\rho(A^*A)^{-1}$ belongs to F whenever $\rho(A^*A)\neq 0$ and $A\in\mathfrak{A}$. We denote by $F^\perp$ the set of operators $A\in\mathfrak{A}$ such that $\rho(A)=0$ for all $\rho\in F$. If $\mathfrak{I}\subset\mathfrak{A}$, $\mathfrak{I}^\perp$ shall denote the set of states ρ such that $\rho(A)=0$ for all $A\in\mathfrak{I}$. E. Effros [2] has shown that the map $\mathfrak{I}\rightarrow\mathfrak{I}^\perp$ is an order inverting bijection between uniformly closed two-sided ideals of $\mathfrak{A}$ and w^*-closed invariant faces of $S(\mathfrak{A})$. Moreover, $(\mathfrak{I}^\perp)^\perp=\mathfrak{I}$, and $(F^\perp)^\perp=F$ when F is a w^*-closed invariant face. If $\mathfrak{I}$ and $\mathfrak{F}$ are uniformly closed two-sided ideals in $\mathfrak{A}$ then $(\mathfrak{I}\cap\mathfrak{F})^\perp=\operatorname{conv}(\mathfrak{I}^\perp,\mathfrak{F}^\perp)$, the convex hull of $\mathfrak{I}^\perp$ and $\mathfrak{F}^\perp$, and $(\mathfrak{I}+\mathfrak{F})^\perp=\mathfrak{I}^\perp\cap\mathfrak{F}^\perp$. If A is a self-adjoint operator in $\mathfrak{A}$ let $\hat{A}$ denote the w^*-continuous affine function on $S(\mathfrak{A})$ defined by $\hat{A}(\rho)=\rho(A)$. It has been shown by R. Kadison, [3] and [4], that the map $A\rightarrow\hat{A}$ is an isometric order-isomorphism of the self-adjoint part of $\mathfrak{A}$ onto all w^*-continuous real affine functions on $S(\mathfrak{A})$. Moreover, if $\mathfrak{I}$ is a uniformly closed two-sided ideal in $\mathfrak{A}$, and ψ is the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{I}$, then the map $\rho\rightarrow\rho\circ\psi$ is an affine isomorphism of $S(\mathfrak{A}/\mathfrak{I})$ onto $\mathfrak{I}^\perp$. Thus the map $\psi(A)\rightarrow\hat{A}\,|\,\mathfrak{I}^\perp$ is an order-isomorphic isometry on the self-adjoint operators in $\mathfrak{A}/\mathfrak{I}$. We shall below make extensive use of these facts. For other references see [1, §1].

THEOREM. *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{I}$ and $\mathfrak{F}$ are uniformly closed two-sided ideals in $\mathfrak{A}$ then*

$$(\mathfrak{I}+\mathfrak{F})^+=\mathfrak{I}^++\mathfrak{F}^+.$$

In order to prove the theorem we may assume $\mathfrak{A}$ has an identity, denoted by I. We first prove a