REFLEXION SPACES AND HOMOGENEOUS SYMMETRIC SPACES

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1. *A reflexion space* is a set M with a multiplication $\mu: M\times M\to M$, $(x, y)\mapsto x\cdot y$, satisfying the following axioms:

$$(S_1)\qquad x\cdot x = x,$$

$$(S_2)\qquad x\cdot(x\cdot y) = y,$$

$$(S_3)\qquad x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z).$$

Let be $S(x)$: $y\mapsto x\cdot y$ the left multiplication with x in M. This is an involutive map of M onto itself leaving x fixed, which may be interpreted as the reflexion in the point x.

Let $\mathfrak{A}$ be a finite dimensional Jordan algebra and I the set of invertible elements of $\mathfrak{A}$. In general for $x, y\in I$ their product xy is not in I, so I does not inherit a multiplicative structure from $\mathfrak{A}$. However, $x\cdot y=2x(xy^{-1})-x^2y^{-1}$ is invertible ([1]), and the multiplication $x\cdot y$ makes I a reflexion space. Every group is a reflexion space with the new product $x\cdot y=xy^{-1}x$. Every set is a reflexion space with the trivial product $x\cdot y=y$ for all x and y.

A reflexion space M where M is a connected paracompact C^∞-manifold and $\mu: M\times M\to M$ is differentiable is called a *differentiable reflexion space*. The following construction gives examples. Let G be a connected Lie group, σ an involutive automorphism of G and H a subgroup of G lying between the group of all fixed points of σ and its identity component. Then G/H is a homogeneous symmetric space and $G(G/H, H)$ is a principal fibre bundle with base space G/H and structure group H. Let H operate on a connected manifold F on the left and let be $G\times_H F$ the bundle associated with $G(G/H, H)$ with typical fibre F (cf [2]). We denote the equivalence class of $(g, x)\in G\times F$ in $G\times_H F$ by $g\otimes x$. In case F is a point, we have $G\times_H F=G/H$.

PROPOSITION 1. *$G\times_H F$ is a differentiable reflexion space with the multiplication*

$$(f\otimes x)\cdot(g\otimes y) = (f(f^\sigma)^{-1}g^\sigma)\otimes y.$$

[1] This work is a generalization of part of the author's doctoral dissertation at the University of Munich under Professor M. Koecher.