AN APPLICATION OF THE CORONA THEOREM TO SOME RINGS OF ENTIRE FUNCTIONS

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1. Introduction. The ring E of entire functions has been extensively investigated in recent years, and a good deal of information on the ideal theory of this ring is now available. The fundamental result here is the theorem of Helmer [3], which asserts that every finitely generated ideal of E is a principal ideal of E. For subrings of E, however, particularly for those determined by growth conditions, this result need no longer hold, and our knowledge of the ideal theory of such rings is quite fragmentary. In this paper we consider some aspects of the ideal theory of certain rings of entire functions defined by growth restrictions on the maximum modulus. For simplicity we restrict the discussion to the ring E_0 of entire functions of exponential type. However, as mentioned below, analogous results hold in more general rings of entire functions. Full details will appear elsewhere.

Recall that an entire function f is of exponential type if there exist constants $A>0$ and $B>0$ such that $|f(z)| \leqq A \cdot \exp(B|z|)$ for all z. It is easy to construct finitely generated ideals of E_0 which are not principal. In fact, there exist functions f, $g \in E_0$ which have no common zeros but which generate a proper ideal of E_0, and here we deal with this latter phenomenon.

MAIN THEOREM. *Let $f_1, \cdots, f_n \in E_0$ and let I denote the ideal of E_0 generated by $f_1, \cdots, f_n$. Then $I = E_0$ if and only if there exist constants $\epsilon > 0$ and $A > 0$ such that for all z*

$$(*) \qquad |f_1(z)| + \cdots + |f_n(z)| \geqq \epsilon \cdot \exp(-A|z|).$$

This result is quite similar to the Corona Theorem for the Banach algebra H_∞ of all functions bounded and analytic on the unit disc $\{z: |z| < 1\}$. (See Hoffman [4] and Carleson [2].) Indeed, our proof of this theorem is based on Carleson's Corona Theorem.

CORONA THEOREM. *Let $f_1, \cdots, f_n \in H_\infty$ and let I be the ideal of H_∞ generated by $f_1, \cdots, f_n$. Then $I = H_\infty$ if and only if there exists a constant $\delta > 0$ such that $|f_1(z)| + \cdots + |f_n(z)| \geqq \delta$, $|z| < 1$. In this case there exist functions $g_1, \cdots, g_n \in H_\infty$ such that $f_1 g_1 + \cdots + f_n g_n = 1$ and such that*

$$\|g_k\|_\infty \leqq \left[\operatorname*{Max}_{1 \leqq j \leqq n} \|f_j\|_\infty \right] \cdot K_1 \delta^{-K_2}, \qquad k = 1, \cdots, n,$$