ON ČEBYŠEV SUBSPACES AND UNCONDITIONAL BASES IN BANACH SPACES

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1. Introduction. Let E be a Banach space, Z a linear subspace of E and x an element of E. An element $z_0 \in Z$ is a *best approximation of x from Z* provided

$$\|x - z_0\| = \inf_{z \in Z} \|x - z\|.$$

Thus, to every linear subspace $Z \subset E$ and element $x \in E$ there corresponds a bounded closed convex (possibly empty) set

$$B_Z(x) = \{z_0 \in Z: \|x - z_0\| = \inf_{z \in Z} \|x - z\|\}.$$

Following Phelps [9] we say that $Z \subset E$ is a Čebyšev subspace if $B_Z(x)$ is one pointed for each $x \in E$.

If (x_i, f_i) is a Schauder basis for E, i.e. $(x_i) \subset E$, $(f_i) \subset E^*$, $f_i(x_j) = \delta_{ij}$ and $x = \sum_{i=1}^{\infty} f_i(x) x_i$ for each $x \in E$, let $L_n = [x_i | i \leq n]$, the linear span of $x_1, \cdots, x_n$ and let $L^n = [x_i | i > n]$, the closed linear span of $x_{n+1}, x_{n+2}, \cdots$. Also, let $s_n(x) = \sum_{i=1}^{n} f_i(x) x_i$ and $s^n(x) = \sum_{i=n+1}^{\infty} f_i(x) x_i = x - s_n(x)$. V. N. Nikol'skiĭ [7], [8] has shown that in a Banach space E with a Schauder basis, an equivalent norm can be given E such that, with respect to this new norm, both L_n and L^n are Čebyšev subspaces and, moreover,

$$B_{L_n}(x) = \{s_n(x)\} \qquad \text{and} \qquad B_{L^n}(x) = \{s^n(x)\}.$$

Now let (x_i, f_i) be an unconditional basis for E, i.e. a Schauder basis with the property that $x = \sum_{i=1}^{\infty} f_{p(i)}(x) x_{p(i)}$ for each permutation p of ω (the positive integers) and each $x \in E$. If $\sigma \in \Sigma$, the finite subsets of ω, let $L_\sigma = [x_i | i \in \sigma]$, $L^\sigma = \{x_i | i \in \omega \backslash \sigma]$, $s_\sigma(x) = \sum_{i \in \sigma} f_i(x) x_i$ and $s^\sigma(x) = x - s_\sigma(x)$. Also, let $B_\sigma(x) = B_{L_\sigma}(x)$ and $B^\sigma(x) = B_{L^\sigma}(x)$.

Motivated by the fundamental work of Nikol'skiĭ mentioned above and by a theorem of Gelfand [3], Singer [10] showed that the norm, $| \ |$, defined by

$$|x| = \sup_{\sigma \in \Sigma} \left\| \sum_{i \in \sigma} f_i(x) x_i \right\| + \sum_{i=1}^{\infty} \|f_i(x) x_i\| / 2^i$$

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