LIE ALGEBRAS ASSOCIATED WITH GENERALIZED CARTAN MATRICES

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1. **Introduction.** If $\mathfrak{L}$ is a finite-dimensional split semisimple Lie algebra of rank n over a field Φ of characteristic zero, then there is associated with $\mathfrak{L}$ a unique $n\times n$ integral matrix (A_{ij})—its Cartan matrix—which has the properties

M1. $A_{ii} = 2, \quad i = 1, \cdots, n,$

M2. $A_{ij} \leqq 0, \quad \text{if } i \neq j,$

M3. $A_{ij} = 0, \quad \text{implies } A_{ji} = 0.$

These properties do not, however, characterize Cartan matrices.

If (A_{ij}) is a Cartan matrix, it is known (see, for example, [**4**, pp. VI-19–26]) that the corresponding Lie algebra, $\mathfrak{L}$, may be reconstructed as follows: Let $e_i, f_i, h_i, i=1, \cdots, n$, be any $3n$ symbols. Then $\mathfrak{L}$ is isomorphic to the Lie algebra $\tilde{\mathfrak{L}}((A_{ij}))$ over δ defined by the relations

$$\left.\begin{aligned} [h_ih_j] &= 0, \\ [e_if_j] &= \delta_{ij}h_i, \\ [e_ih_j] &= A_{ji}e_i, \quad [f_ih_j] = -A_{ji}f_i, \end{aligned}\right\}\text{for all } i \text{ and } j$$

$$\left.\begin{aligned} e_i(\operatorname{ad} e_j)^{-A_{ji}+1} &= 0, \\ f_i(\operatorname{ad} f_j)^{-A_{ji}+1} &= 0, \end{aligned}\right\}\text{if } i \neq j.$$

In this note, we describe some results about the Lie algebras $\tilde{\mathfrak{L}}((A_{ij}))$ when (A_{ij}) is an integral square matrix satisfying M1, M2, and M3 but is not necessarily a Cartan matrix. In particular, when the further condition of §3 is imposed on the matrix, we obtain a reasonable (but by no means complete) structure theory for $\tilde{\mathfrak{L}}((A_{ij}))$.

2. **Preliminaries.** In this note, Φ will always denote a field of characteristic zero. An integral square matrix satisfying M1, M2, and M3 will be called a *generalized Cartan matrix*, or *g.c.m.* for short. Z will denote the integers, and in any Lie algebra we will use the symbol $[l_1, l_2, \cdots, l_n]$ to denote the product $[\cdots[[l_1l_2]\cdots]l_n]$.

[1] These results were obtained in my dissertation at the University of Toronto under the supervision of Professor M. J. Wonenburger.