SINGULAR INTEGRAL OPERATORS ON THE UNIT CIRCLE[1]

BY JOEL DAVID PINCUS

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THEOREM 1. *Let U be unitary and of simple spectral multiplicity and let V be a bounded symmetric operator such that $UV-VU=e(\cdot,\ U^*e)$ where e is cyclic for U. Then V is unitarily equivalent to the operator L defined by*

$$Lx(\tau) = D(\tau)x(\tau) + \frac{1}{\pi i} \mathrm{P}\!\!\int_{\sigma(U)} \frac{k(\tau)k^*(t)}{t-\tau}\, x(t)dt$$ [2]

where $D(\tau)$ is an essentially bounded real-valued function defined on $\sigma(U)$, the spectrum of U, and $k(\tau)$ is an essentially bounded measurable complex-valued function.

We confine ourselves without essential restriction to the case that $k(t)\neq 0$ almost everywhere on $\sigma(U)$.

Let

$$A(l, z) = \exp \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, g(\nu, e^{i\theta})\, \frac{d\nu}{\nu - l}\, d\theta$$

where

$$g(\nu, e^{i\theta}) = \frac{1}{\pi} \arg \frac{D(e^{i\theta}) - \nu - io - |k(e^{i\theta})|^2}{D(e^{i\theta}) - \nu - io + |k(e^{i\theta})|^2}\,.$$

LEMMA 1.

$$[A(\bar{l}, \bar{z})]^{-1} = A^*\left(l, \frac{1}{z}\right)\cdot$$

LEMMA 2. *Let*

$$\phi(\nu, z) = i \exp \int_{-\pi}^{\pi} g(\nu, e^{i\theta})\, \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\theta$$

for $|z|<1$. Then there exists a one-parameter family of positive singular measures, $d\sigma_\nu(\cdot)$, of finite total mass for almost all ν, and a real-valued function $\beta(\nu)$ such that

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[2] The complex conjugate of a function T is denoted by T^*.