BOUNDS FOR LINEAR FUNCTIONALS[1]

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We shall obtain upper and lower bounds for certain functionals associated with linear equations involving positive operators. Attention is focused on these functionals because of their considerable physical significance in applications. A bound from one side is furnished by the usual variational principle. For boundary value problems the reciprocal variational principle introduced by Friedrichs, and later modified by Diaz, provides a complementary bound. In the present article we extend these ideas to an integral equation over a domain E. Our procedure requires information (which is often available) for the same integral equation over some larger domain E'. This approach bears resemblance to the one used by Weinstein and Aronszajn in a series of papers dealing with eigenvalue problems.

Suppose then that we wish to estimate

$$I = \int_E f(x)u(x)dx$$

where

$$(1) \qquad u(x) + \int_E k(x, y)u(y)dy = f(x), \qquad x \in E.$$

We assume that we know how to solve the integral equation

$$(2) \qquad Az = z(x) + \int_{E'} k(x, y)z(y)dy = h(x), \qquad x \in E',$$

for some domain $E' \supset E$.

The situation described above occurs frequently in applications. For instance, if the domains are one-dimensional and the kernel is a difference kernel $k(x-y)$, then the integral equation (2) is easily solved if (a) $k(x)$ has period T and E' is an interval of length T, or (b) $k(x)$ is Fourier transformable and E' is the whole real axis.

Since the method we employ is not restricted to integral equations, we describe it in a slightly more abstract setting.

Let A be a real, self-adjoint, positive operator on the space of real L_2 functions over E'. The usual inner product of two functions $v(x)$

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