DEFORMATIONS OF HOMOMORPHISMS OF LIE GROUPS AND LIE ALGEBRAS

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The purpose of this note is to announce several results on deformations of homomorphisms of Lie groups and Lie algebras. Our main theorems are precise analogues of two basic theorems on deformations of complex analytic structures on compact manifolds, the rigidity theorem of Frölicher-Nijenhuis [3] and the local completeness theorem of Kuranishi [8]. In our results, sheaf cohomology is replaced by the cohomology of Lie groups and Lie algebras. Our proofs rely heavily on the theory of deformations in graded Lie algebras (GLA's) developed in [9]. Our results on Lie algebra homomorphisms follow immediately from the results given there, once the appropriate GLA is defined. Detailed proofs of the results on Lie group homomorphisms (which are only outlined here) will appear elsewhere.

1. Deformations of homomorphisms of Lie algebras. Let $\mathfrak{g}$ and $\mathfrak{h}$ be finite-dimensional real Lie algebras. For each integer $n \geqq 0$, let E^n be the vector space of all alternating n-linear maps of $\mathfrak{g}$ into $\mathfrak{h}$; let $E = \oplus_n E^n$. We define a product on E, also denoted $[\ ,\ ]$ as follows: If $\phi \in E^m$ and $\psi \in E^n$, then $[\phi, \psi] \in E^{m+n}$ is given by

$$[\phi, \psi](x_1, \cdots, x_{m+n}) = \sum_{\sigma} \operatorname{sgn}(\sigma)[\phi(x_{\sigma(1)}, \cdots, x_{\sigma(m)}), \psi(x_{\sigma(m+1)}, \cdots, x_{\sigma(m+n)})],$$

where the sum is taken over all permutations σ of $\{1, \cdots, m+n\}$ such that $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$. When $\phi = \omega \otimes u$ and $\psi = \pi \otimes v$, where ω and π are alternating (real-valued) forms on $\mathfrak{g}$ and $u, v \in \mathfrak{h}$, then $[\phi, \psi] = (\omega \wedge \pi) \otimes [u, v]$. Thus it follows easily that the product $(\phi, \psi) \to [\phi, \psi]$ defines E as a GLA. We define a homogeneous linear map $D: E \to E$ of degree 1 as follows: if $\phi \in E^m$, then

$$(D\phi)(x_1, \cdots, x_{m+1}) = \sum_{i<j} (-1)^{i+j} \phi([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{m+1}).$$

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