MEASURE-THEORETIC UNIFORMITY

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Here we present the principal ideas and results of [5] with some indications of proof. We introduce the notion of measure-theoretic uniformity, and we describe its use in recursion theory, hyperarithmetic analysis, and set theory. In recursion theory we show that the set of all sets T such that the ordinals recursive in T are the recursive ordinals has measure 1. In set theory we obtain all of Cohen's independence results [1], [2] without any use, overt or concealed, of his method of forcing or his notion of genericity. Solovay [8], [9] has extended Cohen's method by forcing statements with closed, measurable sets of conditions rather than finite sets of conditions; in this manner he exploits forcing and genericity to prove: if ZF is consistent, then $ZF+$"there exists a translation-invariant, countably additive extension of Lebesgue measure defined on all sets of reals" + "the countable axiom of choice" is consistent. Solovay's result is also a consequence of the notion of measure-theoretic uniformity.

We begin with the simplest possible example of measure-theoretic uniformity. Let T be an arbitrary set of natural numbers, and let P be the power set of the natural numbers. We think of P as the product of countably many copies of a two-point set $\{a, b\}$. We assign the unbiased measure: $m(\{a, b\})=1$, $m(\{a\})=m(\{b\})=\frac{1}{2}$, and $m(\phi)=0$. We give P the induced product measure denoted by u.

Let $R(T, x, y)$ be a recursive predicate of the set-variable T and the number variables x and y. A familiar uniformity can be expressed as follows: If for some given T we have $(x)(Ey)R(T, x, y)$, then there exists a function f recursive in the given T such that $(x)R(T, x, f(x))$. Before we introduce the measure-theoretic counterpart of this uniformity, we must shift our point of view from Skölem functions to bounding functions in order to make the measure come out right: if for some given T we have $(x)(Ey)R(T, x, y)$, then there exists a function f recursive in T such that $(x)(Ey)_{y\leq f(x)}R(T, x, y)$. Note that the existence of a Skölem function is equivalent to the existence of a bounding function. It is not hard to verify: if $\{T \mid (x)(Ey)R(T, x, y)\}$ has measure 1, then $\{T \mid (Ef)(f \text{ recursive and } (x)(Ey)_{y\leq f(x)}R(T, x, y))\}$ has measure 1. Thus the restriction of the bounding function f to the

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