CRITICAL SUBMANIFOLDS OF DIFFERENTIABLE MAPPINGS

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1. The problems and definitions. There is a general type of problem which contains critical point theory at one extreme, and immersion theory at another. The problems of interest to us lie between these two theories. A glance into their nature is afforded by a simple example to be given following some definitions. Let N^n and M^m denote two differentiable manifolds-with-boundary (perhaps empty) of dimensions n and m respectively, and let $f: N \to M$ be a continuous function with sufficient differentiability at any stage to allow the discussion to proceed. The *deficiency* of f at a point x of N is defined by (minimum (n, m)-rank f at x). Then x is said to be an *ordinary point* of f if f has deficiency zero at x; otherwise x is called a *critical point* of f. If each point of N is an ordinary point of f, we shall simply say f is *ordinary*. Note that if f is ordinary and $n \leqq m$ then f is just an immersion, while if $n \geqq m$ then (in terms of suitable coordinate systems) f is locally a projection.

To proceed with the example, let S^n denote the unit sphere in the $(n+1)$-dimensional euclidean space R^{n+1}, and consider the map $f: S^n \to R^r$ (induced in this instance by the natural projection $R^{n+1} \to R^r$), $r \leqq n$. Then we observe that: (a) the set of critical points of f is confined to the submanifold S^{r-1} of S^n; (b) $f | (S^n - S^{r-1})$, the restriction of f to the complement of S^{r-1} in S^n, and $f | S^{r-1}$ are ordinary; and (c) there exists a map $g: R^r \to R$ (here the natural projection $R^r \to R^1$) such that gf and $(gf) | S^{r-1}$ are Morse functions having the same number of critical points. Now if one attempts to replace S^n in the above by a compact manifold N^n and S^{r-1} by a submanifold K of N, one is immediately faced with the questions of which pairs (N, K) are admissible and what types of singularities to expect? Should it be possible to find an $f: N \to R^r$ satisfying the modified (a) and (b), $N-K$ must for instance admit r linearly independent vector fields and K must be immersible in R^r; while the addition of (c) would require that the Euler characteristics of K and N be congruent modulo two, since the number of critical points of a Morse function defined on a compact manifold is congruent modulo two to the Euler characteristic. These are some aspects of problems which we consider.

In this paper we give a condition of a local nature for the set of critical points of f in the deficiency 1 case to be (not just to be con-