DEDUCTION-PRESERVING "RECURSIVE ISOMORPHISMS" BETWEEN THEORIES[1,2]

BY MARIAN BOYKAN POUR-EL AND SAUL KRIPKE

Communicated by D. Scott, August 2, 1966

Introduction. In this work we discuss recursive mappings between theories which preserve deducibility, negation and implication. Roughly, we prove that any two axiomatizable theories containing a small fragment of arithmetic—this can be stated precisely—are "isomorphic" by a *primitive* recursive function mapping sentences onto sentences which also preserves deducibility, negation and implication (and hence theoremhood, refutability and undecidability). Also we prove between any two effectively inseparable theories formulated as applied predicate calculi there exists a "recursive isomorphism" preserving deducibility, negation and implication. In general, we cannot replace "recursive" by "primitive recursive" in the last result. From this we obtain a classification of all effectively inseparable theories into $\aleph_0$ equivalence classes. The unique maximal element is the equivalence class of those theories containing the small fragment of arithmetic referred to above. A more precise and detailed summary of the results—which answers some questions left open by Pour-El [4]—is given below following some notational remarks.

We believe that interest in the preservation of sentential connectives—especially implication—can be justified as follows. The preservation of implication implies the preservation of modus ponens and modus ponens is closely related to the deductive structure of the theories.

All theories considered in this paper will contain the propositional calculus. For definiteness we assume that implication and negation are the sole primitive propositional connectives: $A \vee B$ is an abbreviation for $\neg A \rightarrow B$; $A \cdot B$ is an abbreviation for $\neg(\neg A \vee \neg B)$. Furthermore in every section except section B the theories discussed will be formulated as applied predicate calculi. All theories considered will be both consistent and axiomatizable.

---

[1] The work of M. B. Pour-El was supported by NSF GP 1612. Results A, C and parts of B were obtained independently by both authors. The remaining results were obtained by the first-named author.

[2] The authors would like to thank Professor Kurt Gödel for his great interest in these results.