ON THE SUMMABILITY OF THE DIFFERENTIATED FOURIER SERIES

BY DANIEL WATERMAN[1]

Dedicated to Professor A. Zygmund on the occasion of his 65th birthday

Communicated by H. Helson, July 21, 1966

A classical theorem of Fatou [2, p. 99] asserts that if $f \in L(0, 2\pi)$ and the symmetric derivative of f at x_0,

$$f_s'(x_0) = \lim_{h \to 0} [f(x_0 + h) - f(x_0 - h)]/2h$$

exists, then the differentiated Fourier series of f is Abel summable to $f_s'(x_0)$ at x_0, or equivalently, if $u(r, x) = a_0/2 + \sum (a_k \cos kx + b_k \sin kx) r^k$ is the associated harmonic function, then

$$\lim_{r \to 1-0} u_x(r, x_0) = f_s'(x_0).$$

Let us suppose that ϕ is a real nonnegative function on an interval to the right of the origin, that $\phi(0) = 0$, and that $\phi(t) = O(t)$ as $t \to 0$. We say that a set is ϕ-dense at a point p if

$$m(E^c \cap I)/\phi(m(I)) \to 0$$

as $m(I) \to 0$, I an interval containing p. If ϕ is the identity function, this reduces to ordinary metric density. In the case $\phi(t) = t^\alpha$, we will say that E is α-dense at p. Proceeding in a manner entirely analogous to the classical definition of approximate limit and derivative, we say that

$$\phi\text{-}\lim_{t \to t_0} {}_{ap}\, g(t) = a$$

if for every $\epsilon > 0$, $E_3 = \{t \mid |g(t) - a| < \epsilon\}$ is ϕ-dense at t_0, and we define the ϕ-approximate symmetric derivative,

$$\phi\text{-}f_{aps}'(x_0) = \phi\text{-}\lim_{h \to 0} {}_{ap}\, [f(x_0 + h) - f(x_0 - h)]/2h.$$

We restrict our attention here to the case of most immediate interest, α-density, and prove the following

THEOREM. *Suppose f is in $L(0, 2\pi)$, of period 2π, essentially bounded in a neighborhood of x_0, and, for some $\alpha \geqq 2$, $y = \alpha\text{-}f_{aps}'(x_0)$. Then the*

[1] Supported by National Science Foundation Grant No. GP-3987.