MULTIPLICATION IN GROTHENDIECK RINGS OF INTEGRAL GROUP RINGS

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Communicated by I. Reiner, June 24, 1966

1. **Introduction.** Let G be a finite group, Z the ring of rational integers, and form the Grothendieck ring $K^0(ZG)$ of the integral group ring ZG. Swan [4] has described multiplication in $K^0(ZG)$ when G is cyclic of prime power order. The purpose of this note is to present results which describe multiplication in $K^0(ZG)$ when G is cyclic or elementary abelian. Full details will appear elsewhere.

Let Q denote the rational field, and recall that the elements of $K^0(QG)$ are Z-linear combinations of symbols $[M^*]$, where M^* ranges over all finitely-generated left QG-modules, and similarly for $K^0(ZG)$. We define a ring epimorphism $\theta\colon K^0(ZG) \to K^0(QG)$ by $\theta[M] = [Q \otimes_Z M]$, and call any linear mapping $f\colon K^0(QG) \to K^0(ZG)$ such that $\theta f = 1$ a *lifting map* for $K^0(ZG)$. Since the Jordan-Hölder Theorem holds for QG-modules, $K^0(QG)$ is the free abelian group with basis $\{[M_i^*]: 1 \leqq i \leqq m\}$, where $\{M_i^*: 1 \leqq i \leqq m\}$ is a full set of nonisomorphic irreducible QG-modules. Swan [4] has shown that to describe multiplication in $K^0(ZG)$ it suffices to describe the products $f[M_i^*] \cdot f[M_j^*]$, for $1 \leqq i, j \leqq m$, and $f[M_i^*]x$, for $1 \leqq i \leqq m$ and $x \in \ker \theta$.

2. **Statement of results.** Let G be cyclic of order n with generator g. For each s dividing n, ζ_s will denote a primitive sth root of unity, and Z_s will denote the ZG-module $Z[\zeta_s]$ on which g acts as ζ_s. Similarly, Q_s will denote the QG-module $Q(\zeta_s)$. Then $K^0(QG)$ is the free abelian group with basis $\{[Q_s]: s \mid n\}$, and $f\colon K^0(QG) \to K^0(ZG)$ by $f[Q_s] = [Z_s]$ is a lifting map. Swan [4] has shown that f is a ring homomorphism. Also, for each s dividing n, G_s will denote the quotient group of G of order s, and if $t \mid s$, $N_{s/t}$ will denote the norm from Q_s to Q_t. By the results of Heller and Reiner [2],

$$\ker \theta = \left\{ \sum_{s \mid n} ([A_s] - [Z_s]) \colon A_s = Z_s\text{-ideal in } Q_s \right\}.$$

THEOREM 1. *Multiplication in $K^0(ZG)$ is given by the formula*

$$[ZG_r]([A_s] - [Z_s]) = \sum_d ([N_{s/s'}(A_s)Z_d] - [Z_d]),$$

for all r, s dividing n, where $s' = s/(r, s)$ and d ranges over all divisors of $[r, s]$ such that $([r, s]/d, s') = 1$.