SUMS OF ULTRAFILTERS

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The main result, an estimate of the cardinality of the set of all ultrafilters producing a given type of ultrafilter (see definition 1.4 and Theorem C in 1.4), is illustrated by a proof of nonhomogeneity of $\beta N - N$ (see 2.1) without using the continuum hypothesis, and by an exhibition of the following two examples.

THEOREM A. *For each positive integer n there exists a space X such that X^n is countably compact but X^{n+1} is not.*

THEOREM B. *There exists a space Y such that each finite product Y^n is countably compact but $Y^{\aleph_0}$ is not.*

By a space we mean a separated uniformizable topological space; and Z^m stands for the product of any constant family $\{Z \mid a \in A\}$ such that the cardinal of A is m.

In our examples the spaces X^{n+1} in A and $Y^{\aleph_0}$ in B are not pseudocompact. An exhibition of A and B with countably compact replaced by pseudocompact is done in [3]; it does not require Theorem C. Trivial examples of spaces with properties in A and B do not seem to be available.

Observe the proof of A and B reduces to the following.

THEOREM A′. *For each positive integer n there exist spaces $X(1), \cdots, X(n+1)$ such that the product of any family $\{X(k_i) \mid i=1, \cdots, n\}$ is countably compact but the product $\{X(j) \mid j \leqq n+1\}$ is not countably compact.*

THEOREM B′. *There exists a sequence $\{Y(j)\}$ of spaces such that the product of any finite subfamily is countably compact but the product of $\{Y(j)\}$ is not.*

Indeed, for an X in A take the sum of spaces $X(j)$ with properties in A′. For Y in B take a one-point countable-compactification of the sum of a sequence $\{Y(j)\}$ with properties in B′; then the product of $\{(j) \times Y(j)\}$ is a closed subspace of Y.

REMARK. In addition, we shall exhibit $\{Y(j)\}$ such that the product of any proper subfamily (e.g. $\{Y(j) \mid j \geqq 2\}$) is countably compact. On the other hand there exists a sequence $\{Y(j)\}$ such that the product of a subfamily is countably compact if and only if the subfamily is finite.