A GENERAL WEDDERBURN THEOREM

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Communicated by S. Smale, May 27, 1966

Let R be a ring, E a left R-module, and set

$$R'(E) = R' = \mathrm{End}_R\, E, \qquad R''(E) = R'' = \mathrm{End}_{R'}\, E.$$

We say that E is *balanced* in case the natural homomorphism $\lambda: R \to R''$ under which $x \mapsto \lambda_x$ where $\lambda_x(v) = xv$, $\forall v \in E$, is an isomorphism. The classical Wedderburn theorem gives a criterion for a module to be balanced. We give a very short proof of a theorem of Morita (in the terminology of [1]) which implies many such criteria.

A left R-module E is said to be a *generator* (for left R-modules) if every R-module can be expressed as a homomorphic image of (possibly infinite) direct sum of copies of E.

THEOREM 1. *Let E be a generator. Then E is balanced.*

PROOF. We first prove that for any module F, $R \oplus F$ is balanced. Given $v \in R \oplus F$, there exists an element $\phi \in R'(R \oplus F)$ such that $\phi(1) = v$ (we view R and F as embedded in $R \oplus F$ as $R \oplus 0$ and $0 \oplus F$ respectively). Let $p: R \oplus F \to R$ be the projection. Let $f \in R''(R \oplus F)$. Then $f(1) = fp(1) = pf(1)$. Hence $f(1) \in R$. It follows that

$$f(v) = f\phi(1) = \phi f(1) = \phi(f(1)\cdot 1) = f(1)\phi(1) = f(1)v.$$

This proves what we wanted.

Let E be a generator. There exists a surjective homomorphism $E^n \to R$ for some integer $n \geqq 1$ (we can take n finite because R is generated by one element). Since R is in fact free, we can write $E^n = R \oplus F$ for some module F. Hence E^n is balanced. We conclude the proof with the following lemma.

LEMMA. *If E is any module and E^n is balanced, then E is balanced.*

PROOF. An element $\phi \in \mathrm{End}_R(E^n)$ can be represented by a matrix (ϕ_{ij}) with $\phi_{ij} \in \mathrm{End}_R(E)$, namely for $v \in E^n$ with components $v_j \in E$ we have

$$\phi(v) = \begin{pmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \phi_{n1} & \cdots & \phi_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix}.$$

Let $f \in R''(E)$. Then the matrix