THE NATURAL METRIC ON SO(n)/SO($n-2$) IS THE MOST SYMMETRIC METRIC

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1. **Introduction.** A smooth manifold M admits many different Riemannian structures. The "symmetry" of a fixed Riemannian structure ν on M is usually called the group of isometries of ν, and denoted by ISO(ν). It was proved by S. B. Myers and N. E. Steenrod [8] that the group of isometries, ISO(ν), is always a Lie group that acts differentiably on M. If M is compact then ISO(ν) is also compact. The dimension of ISO(ν) provides a rough measure of *the degree of symmetry of the given structure* ν. One is tempted by examples to hope that a Riemannian metric which arises naturally will be the best one in the sense that it is the most symmetric among all possible Riemannian structures. In other words, the isometry group of the natural metric will have bigger dimension than the isometry groups of all other metrics.

The following classical theorem in Riemannian geometry shows that the *natural* metrics on spheres and projective spaces are, indeed, the most symmetric metrics: [1, p. 239].

"If $\dim M=m$, then $\dim \mathrm{ISO}(\nu)\leqq m(m+1)/2$ for any Riemannian metric ν on M; and $\dim \mathrm{ISO}(\nu)=m(m+1)/2$ if and only if $M=S^m$ or P^m and the metric ν is the natural metric."

However, if we look at those classical homogeneous spaces other than spheres and projective spaces, we shall find that the dimension of the isometry group of the natural metric is far less than the bound provided by the above classical theorem. For example, let $M=V_{n,2}=\mathrm{SO}(n)/\mathrm{SO}(n-2)$, then

$$\dim M=2n-3 \text{ but } \dim \mathrm{ISO}(\nu)=\dim \mathrm{SO}(n)\times \mathrm{SO}(2)=n(n-1)/2+1$$

for the natural metric ν on $V_{n,2}$. Hence the bound provided by the above theorem, $\frac{1}{2}(2n-3)(2n-2)$, is approximately 4 times bigger than the dimension assumed by the natural one, namely $\frac{1}{2}n(n-1)+1$.

The purpose of this paper is to show that "the natural Riemannian metric on $V_{n,2}$ is indeed the most symmetric metric." We state it more precisely as the following.

THEOREM. *Let $V_{n,2}=\mathrm{SO}(n)/\mathrm{SO}(n-2)$ be the second Stiefel manifold, with n odd and >20. Then*

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