ON THE NATURE OF WEAK SOLUTIONS AND SOME ABSTRACT CAUCHY PROBLEMS

BY ROBERT CARROLL[1]

Communicated by F. Browder, June 13, 1966

We give here some "intrinsic" results about abstract linear Cauchy problems in Hilbert space. It is first shown that weak solutions are really strong solutions of another problem and this leads to a new type of uniqueness theorem for weak solutions. Then the operational nature of certain standard hypotheses in existence theory is exhibited and an abstract existence theorem in linked operators is stated. The details will appear in [5].

1. Let H be a Hilbert space and $V(t)$ a measurable family of Hilbert spaces for Lebesgue measure (cf. Dixmier [7]) with $V(t)\subset H$ algebraically and topologically and dense in H. Let $W=\int^{\oplus} V(t)dt$, $0\leqq t\leqq T<\infty$ with scalar product $((u, v))=\int_0^T((u(t),v(t)))_t dt$. For each $t\in[0, T]$ let $a(t,)$ denote a continuous sesquilinear form on $V(t)\times V(t)$ and let $A(t)\in L(V(t), V(t))$ be the associated continuous linear operator defined by $a(t, x, y)=((A(t)x, y))$ for $x, y\in V(t)$. Suppose further that the field of operators $A(t)$ is measurable (cf. [7]) and $|a(t, u, v)|\leqq M\|u\|_t\|v\|_t$ with M independent of t for $u, v\in V(t)$. We shall consider functions $u\in W$ satisfying

$$(1)\quad -\int_0^T (u, v')dt+\int_0^T a(t, u, v)dt+\lambda\int_0^T (u, v)dt=\int_0^T (f, v)dt$$

for all $v\in W$ with $v'\in L^2(H)$ and $v(T)=0$. Here $(,)$ denotes the scalar product in H and all derivatives are taken in $D'(H)$ (cf. [15]); $f\in L^2(H)$ is supposed given. By the nature of differential problems of this type we can always add a λ term as indicated with arbitrarily large λ (cf. [2], [3], [10]). We are dealing with the case $u(0)=0$ for convenience only. In supposing $\operatorname{Re} a(t, u, u)+\lambda|u|^2\geqq C\|u\|_t^2$ for $u\in V(t)$ Lions proves existence in [10]. Under more hypotheses uniqueness theorems are given by Lions in [10], [11]. In the case that $V(t)=V$ is constant and W is $L^2(V)$ the hypotheses already indicated are enough for existence and uniqueness (see Lions [10]—other proofs can be obtained by specializing Browder's more general nonlinear

[1] Research supported in part by the National Science Foundation under Grant GP-4575.