ZEROS AND FACTORIZATIONS OF HOLOMORPHIC FUNCTIONS

BY WALTER RUDIN[1]

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For $N=1, 2, 3, \cdots$ we let U^N denote the Cartesian product of N copies of the open unit disc U. I.e., U^N consists of all $z=(z_1, \cdots, z_N)$ in C^N (the space of N complex variables) with $|z_j|<1$ for $j=1, \cdots, N$. We write U in place of U^1. If $1 \leqq p < \infty$, $H^p(U^N)$ is the space of all holomorphic functions f in U^N for which

$$\sup (1/2\pi)^N \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |f(r_1e^{i\theta_1}, \cdots, r_Ne^{i\theta_N})|^p d\theta_1 \cdots d\theta_N < \infty,$$

the supremum being taken over all choices of $r_1, \cdots, r_N$ such that $0 \leqq r_j < 1$. The pth root of this supremum is defined to be $\|f\|_p$; this gives a Banach space norm. (The boundary behavior of these functions is discussed in Chapter XVII of [**3**].)

The class of all bounded holomorphic functions in U^N is denoted by $H^\infty(U^N)$.

The *zero-set* of a function f defined in U^N is the set of all $z \in U^N$ at which $f(z)=0$.

It is well known that the zero-set of every $f \in H^p(U)$, for any p, is also the zero-set of some $g \in H^\infty(U)$. These zero-sets, in one variable, are completely characterized by the Blaschke condition $\sum(1-|\alpha_i|) < \infty$. For $N>1$ a different phenomenon occurs:

THEOREM A. *There exists a function f, not identically* 0, *such that*
(a) *$f \in H^p(U^2)$ for all $p<\infty$, but*
(b) *if $g \in H^\infty(U^2)$ and if the zero-set of g contains the zero-set of f, then g is identically* 0.

Let us call a subspace S of $H^p(U^N)$ *invariant* if multiplication by the coordinate functions $z_1, \cdots, z_N$ maps S into S. The closed invariant subspaces of $H^p(U)$ are known precisely: they are generated by inner functions [1, pp. 8, 25]. But if we consider the smallest closed invariant subspace of $H^p(U^2)$ which contains the function f of Theorem A we obtain the following:

COROLLARY. *If $1 \leqq p < \infty$, there is a nontrivial closed invariant subspace of $H^p(U^2)$ which contains no bounded function* (*except* 0).

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