ON NONSEPARABLE REFLEXIVE BANACH SPACES

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The purpose of this paper is to show that certain known results concerning separable spaces hold also for nonseparable reflexive Banach spaces. Our main result (Theorem 1) proves a special case of a conjecture of H. H. Corson and the author [1] while the corollary proves some conjectures of V. Klee (see for example [2]). In order to state Theorem 1 we introduce the following notation: Let Γ be a set; by $c_0(\Gamma)$ we denote the Banach space of scalar valued functions f on Γ, such that $\{\gamma; |f(\gamma)| > \epsilon\}$ is finite for every $\epsilon > 0$, with the sup norm.

THEOREM 1. *Let X be a reflexive Banach space. Then there is a one to one bounded linear operator from X into $c_0(\Gamma)$ for a suitable set Γ.*

This theorem was proved in [3] for spaces X which have the metric approximation property (M.A.P.) introduced by Grothendieck. We shall show here how to modify the proof in [3] so that it will not depend on the assumption concerning the M.A.P. As noted in [3] the following corollary is an easy consequence of Theorem 1 and known results.

COROLLARY 1. *Let X be a reflexive Banach space. Then*

(i) *X has an equivalent strictly convex norm.*

(ii) *X has an equivalent smooth norm.*

(iii) *The norm of X is Gateaux differentiable at a dense subset of X.*

(iv) *If K is a bounded closed convex subset of X then K is the closed convex hull of its exposed points.*

We pass to the proof of Theorem 1. It is clearly enough to consider only real spaces. Our first lemma holds for a general Banach space.

LEMMA 1. *Let X be a Banach space and let B be a finite-dimensional subspace of X. Let k be an integer and let $\epsilon > 0$. Then there is a finite-dimensional subspace Z of X containing B such that for every subspace Y of X containing B with* dim $Y/B = k$ *there is a linear operator $T: Y \to Z$ with $\|T\| \leq 1 + \epsilon$ and $Tb = b$ for every $b \in B$.*

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