THE SPECTRAL THEORY OF SELF-ADJOINT WIENER-HOPF OPERATORS[1]

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Communicated by M. Kac, April 5, 1966

Introduction. This note will present a new method for the construction of solutions to the Wiener-Hopf equation

$$Lx(s) \equiv \int_0^\infty k(s-t)x(t)dt = \xi x(s), \qquad 0 \leqq s \leqq \infty. \tag{1}$$

However, in addition to presenting an alternate to the Wiener-Hopf factorization method, we will construct results not obtainable by that method; namely a complete spectral representation of self-adjoint operators of the given form. That is, we will construct a direct integral Hilbert space $\mathcal{H}^*$ which will be characterized in terms of an integer-valued Lebesgue measurable multiplicity function, $m(\xi)$, which we will exhibit explicitly, and an isometric mapping of the basic Hilbert space onto $\mathcal{H}^*$ given explicitly by a sequence of $m(\xi)$ integral operators whose kernels are generalized eigenfunctions of L; furthermore we will exhibit the transformation inverse to $\mathcal{S}$ explicitly as a sum of integral operators acting on the components of $\mathcal{H}^*$.

These results will be obtained through a simple reduction to the author's previous work on Barrier Related Spectral Problems [**1**], [**2**], [**3**], and we will not require, as it is customary in the standard Wiener-Hopf procedure that $k(t) = O(e^{-a|t|})$ for some $a > 0$ or that $k(t)$ is real and continuous except for a finite number of jumps [**4**].[2]

Finally, after exhibiting our method, we will show for a particular standard example how the textbook solutions can be recovered from our formulation.

Assumptions and basic definition. We will require Hermiticity throughout; namely, $k(t) = k(-t)^-$, and, in addition, we require that $k(t)$ and $|k(t)|^2$ be absolutely integrable. For simplicity, we also require that $\hat{k}(\lambda)$, the Fourier transform of $k(t)$, be nonnegative.

Reduction to canonical form. Let P be the orthogonal projection from $L_2(-\infty, \infty)$ onto $L_2(0, \infty)$ such that

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[1] This work was performed under the auspices of the U. S. Atomic Energy Commission.

[2] See [5] however.