ON A CERTAIN INVARIANT OF A LOCALLY COMPACT GROUP

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Group here always means a locally compact Hausdorff group, *subgroup* means a closed subgroup. Let G be a group, H a subgroup and G/H the locally compact homogeneous space of left cosets $\dot{x}=xH$. We denote by $\mathfrak{K}(G)$ $[\mathfrak{K}(G/H)]$ the family of all compact subsets of $G[G/H]$. The group G acts on G/H in a natural way. If $X\subset G$ and $Y\subset G/H$, write XY for the set of all elements $x\dot{y}$, $x\in X$, $\dot{y}\in Y$. Now assume that G/H has a nontrivial invariant positive measure $d\dot{x}$, e.g. the left invariant Haar measure, if H is normal. For a measurable set U in G/H let $|U|$ or $|U|_{G/H}$ be its measure. Then we define:

$$I(G/H) = \sup_{K\in\mathfrak{K}(G)} \inf_{\substack{U\in\mathfrak{K}(G/H)\\ |U|>0}} \frac{|KU|_{G/H}}{|U|_{G/H}}.$$

Evidently $1\leqq I(G/H)\leqq\infty$ and $1=I(G/H)$ if G/H is compact. Let E be the trivial subgroup of order one in G. We identify G and G/E.

For a positive Radon measure μ and a Borel function f on G, the convolution $\mu*f$ is defined as

$$\mu*f(x) = \int_G f(y^{-1}x)\, d\mu(y).$$

If $\mathfrak{F}$ is a set of Borel functions, let $\mu*\mathfrak{F}$ be the set of all $\mu*f$, $f\in\mathfrak{F}$ (if this set is well defined). For $1\leqq p\leqq\infty$, let $\mathfrak{L}^p(G)$ be the usual $\mathfrak{L}^p$-space of the group G. If μ is a positive bounded Radon measure, then $\mu*\mathfrak{L}^p(G)\subset\mathfrak{L}^p(G)$ for all $p\geqq 1$. In [2] I proved a partial converse of this fact, as follows.

Let $p>1$. If $\mu*\mathfrak{L}^p(G)\subset\mathfrak{L}^p(G)$ and $I(G)<\infty$, then μ is bounded.

As I pointed out in [2], this implies the following.

Let $p>1$. If $\mathfrak{L}^p(G)$ is closed under convolution and $I(G)<\infty$, then G is compact.

This latter statement, without the hypothesis $I(G)<\infty$, is the so called $\mathfrak{L}^p$-conjecture, stated and discussed by Żelazko, Rajagopalan and others [3], [4], [5], [6], [7].

The main result of this note is an inequality for $I(G)$, which implies the finiteness of $I(G)$ for a fairly large class of groups. Actually it reduces the problem of checking this finiteness to the case of simple Lie groups and finitely generated discrete groups.