ON SOME QUESTIONS IN NOETHERIAN RINGS

BY LANCE W. SMALL

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1. **Introduction.** Goldie's theorem [1] establishes (among other things) that right Noetherian rings without nilpotent ideals have (right) classical quotient rings. In the general case the author [3] found a necessary and sufficient condition that an arbitrary right Noetherian ring have a right Artinian quotient ring. Using this criterion, it can be shown that a right hereditary, right Noetherian ring has a right Artinian quotient ring [2].

In this note we present an example of:

(1) a right and left Noetherian ring with a right, but not a left quotient ring and

(2) a right and left Noetherian ring with no quotient ring on either side.

2. **Notations and definitions.** From now on ring means ring with unit element, and Noetherian means right and left Noetherian. $N(R)$ denotes the maximal nilpotent ideal of a Noetherian ring R.

DEFINITION. If M is a subset of a ring A, then the *right annihilator* of M, $r(M)$, is $\{a \in A \mid Ma = 0\}$. We shall write $r_A(M)$ if there is a possibility of confusion about the ring. The *left annihilator* of M, $l(M)$, is defined analogously.

Recall that an element a in a ring R is *regular* if $r(a) = l(a) = 0$.

DEFINITION. A right (left) ideal I of a ring R is *essential* if I intersects every nonzero right (left) ideal of R nontrivially.

DEFINITION. $Z_r(R) = \{a \in R \mid r(a) \text{ is essential}\}$ is called the (right) *singular ideal* of R. $Z_l(R)$ is defined analogously.

3. **The examples.** Let $\boldsymbol{Z}$ denote the integers and $p \in \boldsymbol{Z}$ a prime. Define T to be the ring of all two-by-two "matrices" of the form:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \qquad a \in \boldsymbol{Z}, \qquad b \in \boldsymbol{Z}/(p), \qquad c \in \boldsymbol{Z}/(p),$$

where addition is component-wise and multiplication is given by:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}$$

where $\boldsymbol{Z}$ acts on $\boldsymbol{Z}/(p)$ in the usual way.

LEMMA 1. *T is Noetherian.*