CONSTRUCTION OF GLOBALLY CONVERGENT ITERATION FUNCTIONS FOR THE SOLUTION OF POLYNOMIAL EQUATIONS

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Iteration functions for the approximation of zeros of a polynomial P are usually given as explicit functions of P and its derivatives. We introduce a class of iteration functions which are themselves constructed according to a certain algorithm given below. The construction of the iteration functions requires only simple polynomial manipulation which may be performed on a computer.

Let P be a real monic polynomial of degree n with distinct zeros $\rho_1, \cdots, \rho_n$ and let the dominant zero ρ_1 be real. The theory may be extended to multiple zeros, dominant complex zeros, and subdominant zeros.

Let $B(t)$ be an arbitrary polynomial of degree at most $n-1$ with $B(\rho_1)\neq 0$. Define a sequence of polynomials of degree $n-1$ by

$$G(0, t, B) = B(t), \quad G(\lambda + 1, t, B) = tG(\lambda, t, B) - \alpha_0(\lambda)P(t), \qquad \lambda = 0, 1, \cdots,$$

where $\alpha_0(\lambda)$ is the leading coefficient of $G(\lambda, t, B)$. From the polynomial $G(\lambda, t, B) \equiv G_1(\lambda, t, B)$, form the polynomial $G_p(\lambda, t, B)$ for any positive integer p by

$$G_p(\lambda, t, B) = \sum_{k=0}^{p-1-k} [-P]^{p-1-k} \frac{G^{(p-1-k)}(\lambda, t, B)}{(p-1-k)!} V_k(t),$$

where $V_k(t)$ is formed by

$$V_0(t) = 1, \quad V_k(t) = P'(t)V_{k-1}(t) - \frac{P(t)}{k} V'_{k-1}(t).$$

Define an iteration function for *fixed* p and λ by

$$\phi_p(\lambda, t, B) = t - P(t) \frac{G_{p-1}(\lambda, t, B)}{G_p(\lambda, t, B)} \cdot$$

The global nature of the convergence is given by

THEOREM. *Let t_0 be an arbitrary point in the extended complex plane such that $t_0 \neq \rho_2, \rho_3, \cdots, \rho_n$ and let $t_{i+1} = \phi_p(\lambda, t_i, B)$. Then for all sufficiently large but fixed λ, the sequence t_i is defined for all i and $t_i \to \rho_1$.*