ON INFINITE INSEPARABLE EXTENSIONS OF EXPONENT ONE

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Communicated by G. D. Mostow June 30, 1965

Let K be a field of characteristic $p\neq 0$ and Der K denote the vector space over K of all derivations of K. A classical theorem of Jacobson [2], strengthened by the author [1], asserts that the subfields L of K with $L\supset K^p$ and $[L:K]$ finite are in natural one-one correspondence with the finite dimensional "restricted" subspaces of Der K, i.e., with those subspaces V such that $\dim_K V<\infty$ and such that $\phi\in V$ implies $\phi^p\in V$; the correspondence associates to L the space $\mathrm{Der}_L\, K$ of all derivations vanishing on L. (It follows that a finite dimensional restricted subspace is necessarily a Lie algebra.) The problem of extending this result after the fashion of Krull to fields $L\supset K^p$ with $[K:L]$ possibly infinite has been raised explicitly (cf. [3, p. 191]) but not answered. The purpose of this note is to show that the obvious conjecture in fact holds.

1. The Krull topology and statement of the main theorem. Let Der K be topologized by taking as a base for the neighborhoods of zero those subspaces V of the form $\mathrm{Der}_L\, K$ with L a finite extension $K^p(x_1, \cdots, x_n)$ of K^p; this will be called the *Krull topology*. The closure of an arbitrary subspace V in the Krull topology will be denoted by $\overline{V}$. Given an arbitrary element ϕ of Der K, the set of all $x\in K$ which are constants for ϕ, i.e., such that $\phi(x)=0$, will be denoted K_ϕ. We shall further denote by D_ϕ the smallest restricted subspace of Der K containing ϕ, and by $\overline{D}_\phi$ its closure.

It is immediate that the closure of a restricted subspace is again restricted, and that a subspace of the form $\mathrm{Der}_L\, K$ is both closed and restricted.

THEOREM. *Let K be a field of characteristic $p\neq 0$. Then the subfields L containing K^p are in natural one-one correspondence with the closed restricted subspaces of* Der K, *the correspondence assigning to L the space* $\mathrm{Der}_L K$. *(It follows that a closed restricted subspace is in particular a Lie algebra.) Further, every closed restricted subspace is of the form $\overline{D}_\phi$ for some ϕ in* Der K.

2. Proof of the theorem. Before the proof we give several lemmas.

[1] The author wishes to acknowledge the support of the National Science Foundation under contract GP-3683.