AXIALLY-SYMMETRIC BOUNDARY-VALUE PROBLEMS[1]

ALBERT E. HEINS

I. Introduction. The axially-symmetric boundary-value problems which we shall discuss are solutions of the elliptic partial differential equations

$$\nabla^2\Phi \pm k^2\Phi = 0 \tag{1.1}$$

where Φ is a function of the conventional cylindrical coordinates (r, ϕ, z) and the boundary conditions are supplied on some surface of revolution S. The term axially-symmetric does not refer to the fact that $\Phi(r, \phi, z)$ is independent of the angular coordinate ϕ, but to the fact that S is a surface of revolution about the z axis. This address will be concerned with a class of boundary-value problems for equation (1.1) for which we supply Dirichlet, Neumann or mixed-type[2] boundary data on S.

A classical method for studying such boundary-value problems has been the use of expansion or integral transform theorems appropriate to the coordinate system of the surface S, at least when they are available. Yet it is has been often observed that there are cases in which the special functions of the coordinate system disappear and we may be left with simple results. For the axially-symmetric geometries we consider here, this phenomenon is not accidental, and indeed, we find that the boundary-value problems of which we speak can be expressed in terms of boundary information of a two-dimensional harmonic function. This information is contained in the Poisson integral representation of solutions of the equations (1.1) [23], [24].[3] Once we supply the boundary data for equation (1.1), we have an integral equation for the two-dimensional harmonic function of which we spoke. In conjunction with the Poisson integral representation, we also have a classical one due to Helmholtz which has a Green's function associated with (1.1) as a kernel. We shall see that the continuation of axis data[4] provided by the Helmholtz representa-

An address delivered before the Evanston meeting of the Society on November 23, 1962 by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings; received by the editors July 5, 1965.

[1] Publication supported by the Air Force Office of Scientific Research.

[2] By mixed-type boundary data we mean that a linear combination of Φ and its normal derivative is assigned on S.

[3] Numbers in brackets refer to the references cited at the end of the paper.

[4] That is, data supplied on the axis of symmetry $r=0$.