A FACTORIZATION THEOREM FOR HOLOMORPHIC FUNCTIONS OF POLYNOMIAL GROWTH IN A HALF PLANE

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A function $F(p)$, $p=\sigma+i\omega$, belongs to class H^+, if it is holomorphic in the half plane $\operatorname{Re} p>0$, and if it is bounded by a polynomial uniformly in every half plane $\operatorname{Re} p\geqq\sigma>0$. The significance of the H^+ class is due to an important theorem of L. Schwartz [1]; this theorem tells us that H^+ characterizes the collection of Laplace transforms of certain distributions in D'_+ (here, as in what follows, we use the terminology and notation of distributions to be found in [2]). Moreover, H^+ is basic in the following extension of an L_2 result of Paley-Wiener.

THEOREM 1 [3]. *A necessary and sufficient condition, in order that $F_\omega\in S'$ (tempered distribution) be the boundary value in the S' topology of an H^+ function $F(\sigma+i\omega)$, as $\sigma\to 0$, is that F_ω be the Fourier transform of some $\hat{F}\in S'\cap D'_+$. In particular, $F(p)$ is the Laplace transform of $\hat{F}$.*

The classical version of Theorem 1 is obtained when S' is replaced by $L_2\subset S'$ and H^+ is replaced by the Hardy class $H^2\subset H^+$. Now a function which is of class H^2, in the right half plane, also admits a factorization into inner and outer factors, as given by a well-known theorem (cf. [4, p. 67]). The inner factor is a.e. of modulus one on the boundary and can be written as the product of a singular function and a Blaschke product taken with the zeros of the H^2 function; the outer factor is nonzero and is itself in H^2. Actually, H^2 is completely characterized by such a factorization.

The purpose of this note is to extend the classical result to the H^+ class by proving the following theorem.

THEOREM 2. *$F(p)\in H^+$ can be factored as $F(p)=p^kB(p)S(p)g(p)$ (where k is some nonnegative integer, $B(p)$ is a convergent Blaschke product formed with the zeros of $F(p)$, $S(p)$ is singular, $g(p)$ is nonzero and in H^+, and $g(p)$ has a D'_{L_2} boundary value taken in the S' topology as $\sigma\to 0$), if and only if $F(p)$ has S' boundary values as $\sigma\to 0$.*

PROOF. Let $F(p)\in H^+$ tend to $F_\omega\in S'$ as $\sigma\to 0$. Then, for sufficiently large m, $F(p)=p^mf(p)$ and $f(p)\to^{S'}f_\omega\in D'_{L_2}$ (cf. [3]). Now let $f_n(\omega)$ be the regularizations of f_ω; then the inverse Fourier transforms $\mathfrak{F}^{-1}f_n\equiv\hat{f}_n\in L_2(0,\ \infty)$, since $f_n\in L_2$. Moreover (Paley-Wiener), the