ASYMPTOTIC VALUES OF HOLOMORPHIC FUNCTIONS OF IRREGULAR GROWTH

BY ALFRED GRAY AND S. M. SHAH[1]

Communicated by Maurice Heins, May 3, 1965

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be holomorphic with radius of convergence R $(0<R\leqq\infty)$, and let $\mu(r)$ denote the maximum term and $\nu(r)$ the central index of $f(z)$. By definition, for $r>0$, $\mu(r)=\max\{|a_n|r^n|n=0,1,2,\cdots\}$ and so $\mu(r)=|a_{\nu(r)}|r^{\nu(r)}$. We shall assume that $\mu(r)\to\infty$ as $r\to R$ and $f(z)$ is not a polynomial. In this note we give a technique for comparing $f(z)$ with its maximum term which shows that, for certain functions $f(z)$ which are of very slow growth, or whose power series have wide gaps, $f(z)$ has no finite asymptotic values. Our result is to be compared with Wiman's theorem [1, Chapter 3], [5]: If $f(z)$ is an entire function of order $\rho<\frac{1}{2}$ then $f(z)$ has no finite asymptotic values. However, the class of functions for which we show the nonexistence of finite asymptotic values is different from that of Wiman; in particular we allow the functions to have a finite radius of convergence.

Let $z=re^{i\theta}$ and define

$$\mu(re^{i\theta}) = \mu(r)e^{i\nu(r)\theta}$$

for $r>0$ and $0\leqq\theta<2\pi$. Then $\mu(z)$ is a complex extension of $\mu(r)$; it is piecewise continuous, but has discontinuities where $\nu(|z|)$ is discontinuous.

Let $\gamma(t)$ be a (continuous) receding curve such that $|\gamma(t)|\to R$ as $t\to\infty$. Then $\gamma(t)$ is an *asymptotic path* of $f(z)$ if as $t\to\infty$, $f(\gamma(t))$ tends to a limit ω, called an *asymptotic value*; analogously with this definition we shall call $\gamma(t)$ a *μ-asymptotic* path if $f(\gamma(t))/\mu(\gamma(t))$ tends to a limit ω as $t\to\infty$, and we say that ω is a *μ-asymptotic value.* For example, e^z has μ-asymptotic value ∞ along the positive real axis, but has μ-asymptotic value 0 along any path to ∞ in any angle which excludes the positive real axis. The following theorem is obvious, since $\mu(r)\to\infty$.

THEOREM 1. *If $\gamma(t)$ is an asymptotic path of $f(z)$ with finite asymptotic value, then $\gamma(t)$ is a μ-asymptotic path of $f(z)$ with μ-asymptotic value* 0.

Next we investigate some situations in which $f(z)$ has no μ-asymp-

[1] The work of this author was supported by NSF grant GP-4311.