ALL 3-MANIFOLDS IMBED IN 5-SPACE

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It has been shown by Hirsch [4, Theorem 4.6] that any nonclosed smooth n-manifold can be imbedded in R^{2n-1} ($n \geqq 1$). Also, Haefliger and Hirsch [2, Theorem 1.1] proved that if $n>4$, a smooth closed n-manifold can be imbedded in R^{2n-1} if and only if its normal Stiefel-Whitney class $\overline{W}^{n-1}$ vanishes (and, according to Massey and Peterson [10, Lemma 9], $\overline{W}^{n-1}$ can only be nonzero if the manifold is nonorientable and n a power of 2). As to the cases $n \leqq 4$, the imbedding is clearly impossible for $n \leqq 1$; it can be done for $n=2$ (here, $\overline{W}^1$ vanishes precisely in the orientable case) classically. In this note, we shall also accomplish the proof for $n=3$. Only $n=4$ remains undecided. We note that the only gap in the proof of [2] for $n=4$ is the appeal to the result of Haefliger; in the piecewise linear case this can be filled by using instead a result of Irwin [6]. (The details of this argument will appear in a paper of Hirsch.) For $n=3$, every piecewise linear manifold can be smoothed, and every paracompact manifold triangulated, so all these are included in our result. The pattern of the proof resembles that in the orientable case (settled by Hirsch in [5]).

All the imbedding theorems for 3-manifolds depend on the following principle, first used by Hirsch in [4].

Suppose M immersed in R^κ, and that M has a subcomplex S^s such that M can be imbedded in any neighbourhood of S, and $\kappa \geqq 2s+1$. Then M can be imbedded in R^κ.

For since imbeddings are dense in the space of maps $S^s \rightarrow R^\kappa$, and immersions are open in the space of maps $M \rightarrow R^\kappa$, we may suppose that the immersion imbeds S. It then imbeds some neighbourhood of S (it is not necessary for this to assume S compact) and the result follows.

We shall call a subcomplex S as above a *spine* of M. Our proposed theorem will then follow from:

Any closed 3-manifold M bounds a 4-manifold W such that (a) *W immerses in R^5, and* (b) *W has a 2-dimensional spine.*

The proof of this will be split in three parts: discussion of the tangent bundle of M (for completeness), cobordism theory (to obtain W satisfying (a)) and surgery (to obtain also (b)).

A priori, the group of the tangent bundle of M may be taken as O_3. We note two subgroups each with two elements: O_1 and the centre Z