THE NORMALITY OF TIME-INVARIANT, SUBORDINATIVE OPERATORS IN A HILBERT SPACE

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1. **Introduction.** Let $(U_t,\ t\in\mathcal{R})$ be a strongly continuous group of unitary operators on a (complex) Hilbert space $\mathcal{H}$ onto $\mathcal{H}$, and E be its spectral measure on the family $\mathcal{B}$ of Borel subsets of the real number field $\mathcal{R}$, so that $U_t=\int_{-\infty}^{\infty}e^{i\lambda t}\,E(d\lambda)$ (Stone's Theorem). Let $\mathcal{S}_x$ be the cyclic subspace generated by x under the action of the U_t, $t\in\mathcal{R}$, or, equivalently, under that of $E(B)$, $B\in\mathcal{B}$. Let L_x be the cyclic projection of x, i.e., the projection on $\mathcal{H}$ with range $\mathcal{S}_x$. Using a term due to Kolmogorov [2, §4] we shall say that x is *subordinate* to y if $\mathcal{S}_x\subseteq\mathcal{S}_y$.

Our purpose is to assert the following theorem and deduce some corollaries which generalize known results:

1.1. THEOREM. *Given a strongly continuous unitary group* $(U_t,\ t\in\mathcal{R})$ *with spectral measure* E *on the family* $\mathcal{B}$ *of Borel subsets of* $\mathcal{R}$, *let* T *be any (single-valued, unbounded) linear operator from* $\mathcal{H}$ *to* $\mathcal{H}$ *such that*

(i) $U_tT=TU_t$ *for all* $t\in\mathcal{R}$ *(*T *is "time-invariant"),*

(ii) $T(x)\in\mathcal{S}_x$ *for all* $x\in\mathfrak{D}_T$ *(*T *is "subordinative"),*

(iii) T *is closed and* $\mathcal{H}_0=\operatorname{clos}\mathfrak{D}_T$ *is separable.*

Then there exists a complex-valued Borel-measurable function ϕ *on* $\mathcal{R}$ *such that*

$$T=\int_{-\infty}^{\infty}\phi(\lambda)\,E_0(d\lambda),\quad \textit{where } E_0=\operatorname{Rstr.}_{\mathcal{H}_0}E.^{2}$$

ϕ *is unique up to sets of zero* E_0*-measure.*

2. **Statistical theory of linear filters.** Theorem 1.1 has its genesis in the statistical theory of linear filters as conceived by N. Wiener. In this theory the signals are realizations of strictly stationary stochastic processes (S.P.). It is assumed that these processes are governed by a single measure-preserving, ergodic flow over a probability space $(\Omega,\ \mathcal{B},\ P)$. This flow induces the unitary group $(U_t,\ t\in\mathcal{R})$ on the Hilbert space $H=L_2(\Omega,\ \mathcal{B},\ P)$. T is the filter transformation; it

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[2] $\operatorname{Rstr.}_D A$ denotes the restriction of the operator A to the subset D of its domain.