TOPOLOGY OF QUATERNIONIC MANIFOLDS

BY VIVIAN YOH KRAINES[1]

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We give here a quaternionic analogue (Theorem 4) of the Hodge decomposition theorem [2, p. 26] for a Riemannian manifold with holonomy group contained in $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$. Applying Chern's theorem in [1] (also [3]), we obtain some consequences on Betti numbers (Theorem 5).

Let $\boldsymbol{K}^n$ denote the n-dimensional vector space over the field $\boldsymbol{K}$ of quaternions, with the inner product

$$(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2}\sum_{i=1}^{n} (p_i\bar{q}_i + q_i\bar{p}_i),$$

where

$$\boldsymbol{p} = (p_1, \cdots, p_n), \qquad \boldsymbol{q} = (q_1, \cdots, q_n) \text{ and}$$
$$p_i = \overset{0}{p_i} + \overset{1}{p_i}i + \overset{2}{p_i}j + \overset{3}{p_i}k,$$
$$q_i = \overset{0}{q_i} + \overset{1}{q_i}i + \overset{2}{q_i}j + \overset{3}{q_i}k$$

are quaternions.

Let $\mathrm{Sp}(n)$ be the set of all endomorphisms, A, of $\boldsymbol{K}^n$, satisfying the identity $(A\boldsymbol{p}, A\boldsymbol{q}) = (\boldsymbol{p}, \boldsymbol{q})$. $\mathrm{Sp}(n)$ is the set of all $n\times n$ matrices preserving the inner product. Then $\mathrm{Sp}(1)$ is the set of all unit quaternions. We define the action of $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ on $\boldsymbol{K}^n$ as follows:

$$(A, \lambda)\boldsymbol{p} = A\boldsymbol{p}\lambda, \quad \text{for } (A, \lambda) \in \mathrm{Sp}(n) \times \mathrm{Sp}(1),$$

i.e., we multiply $\boldsymbol{p}$ on the left by the matrix A and on the right by the unit quaternion λ.

DEFINITION. We define three skew symmetric bilinear forms Ω_I, Ω_J and Ω_K on $\boldsymbol{K}^n$ as follows:

$$\Omega_I(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}i, \boldsymbol{q}),$$
$$\Omega_J(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}j, \boldsymbol{q}) \text{ and}$$
$$\Omega_K(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}k, \boldsymbol{q}).$$

Note that Ω_I, Ω_J and Ω_K may be thought of as exterior 2-forms of $\boldsymbol{K}^n$ considered as a $4n$-dimensional real vector space.

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