DIFFERENTIABLE FUNCTIONS ON CERTAIN BANACH SPACES

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Communicated by G. A. Hunt, October 21, 1964

The main result in this note, Theorem 2, can be thought of as a very strong maximum modulus type theorem. For example, let D be a bounded connected open set in $C(0, 1)$, and let f: $\mathrm{Cl}D \to R^n$ be continuous and differentiable in D. Then f is determined by its values on the boundary of D. More exactly, $f(\mathrm{Cl}D) \subset \mathrm{Cl}f(\partial D)$. More generally, if F is any Banach space and f: $\mathrm{Cl}D \to F$ is completely continuous and differentiable in D, then $f(\mathrm{Cl}D) \subset \mathrm{Cl}f(\partial D)$. Note that these results are false if $C(0, 1)$ is replaced by a Hilbert space.

THEOREM 1. *Let D be a connected bounded open set in l^p where p is not an even integer. Assume f is a real-valued function, continuous on* $\mathrm{Cl}D$ *and n-times differentiable in D with $n \geqq p$. Then $f(\mathrm{Cl}D) \subset \mathrm{Cl}f(\partial D)$.*

This generalizes a result proved in 1954 by Kurzweil [1]. Kurzweil assumed that f was n-times continuously differentiable, that D was a ball $B(x_0, r)$, and showed that $\inf\{|f(x)-f(x_0)| : \|x-x_0\| = r\} = 0$.

COROLLARY 1. *Let f be an n-times differentiable function on l^p, where $n \geqq p$, and p is not an even integer. If f has its support in a bounded set, then f is identically zero.*

In particular, it follows that, for $n \geqq p$, C^n partitions of unity do not exist whenever p is not an even integer. This partially settles a question raised in Lang [2]. It should be noted, however, that this is implied by Kurzweil's result.

COROLLARY 2. *Let E be a Banach space containing a subspace equivalent to l^1. Assume D is a connected bounded open set in E, and that f is a real-valued function continuous on* $\mathrm{Cl}D$ *and differentiable in D. Then $f(\mathrm{Cl}D) \subset \mathrm{Cl}f(\partial D)$.*

$C(0, 1)$ and $L^1(0, 1)$ are examples of spaces where Corollary 2 holds. More generally, any separable Banach space with an unconditional basis and nonseparable dual contains a subspace equivalent to l^1. It may be that any separable Banach space with a nonseparable dual has a subspace equivalent to l^1. Corollary 2 generalizes an unpublished result of Edward Nelson who showed that, in $C(0, 1)$, differentiable functions with bounded support are identically zero.

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[1] Research supported in part by NSF grant GP-1645.