THE SECOND VARIATION FOR VARIATIONAL PROBLEMS IN CANONICAL FORM

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1. **Introduction.** This work is a sequel to [2], although it can be read independently. In the paper on which this announcement is based we develop a differential-geometric formalism for variational problems that can serve as well for multiple as for single integral problems, that does not require the introduction of local coordinate systems (which is often awkward in geometric situations) and that is well-adapted to computation. For example, we compute quite easily the second variation formula for minimal submanifolds of Riemannian spaces (which apparently is not in the literature) and can then present some geometric applications, since the geometric meaning of the terms is very clear in our formula 3.2.

The consideration of variational problems in "canonical form" (see below for the definition) leads to a description of the extremal submanifolds in terms of Cartan's theory of exterior differential systems. We will make use here of the geometric ideas and notation that were introduced in [3] for dealing with Cartan's theory.

2. **The first and second variation.** Let N be an oriented manifold with oriented boundary ∂N. Let M be another manifold, with $\dim N \leqq \dim M$. Let E be the space of submanifold maps of N into M. For $\phi \in E$, the "tangent space" to E at ϕ, denoted by E_ϕ, is a map, typically denoted by v, of $N \to T(M)$ ($=$the tangent bundle of M) such that $v(p) \in M_{\phi(p)}$ for $p \in N$. A deformation $t \to \phi_t$ of ϕ, i.e., a curve in E beginning at ϕ, defines an element $v \in E_\phi$: For $p \in N$, $v(p)$ is the tangent vector to the curve $t \to \phi_t(p)$ at $t=0$. The following formula is proved in [3]:

$$2.1. \qquad \frac{\partial}{\partial \tau} \phi_t^*(\theta) \bigg|_{t=0} = \phi^*(v \urcorner d\theta) + d\phi^*(v \urcorner \theta)$$

for each differential form θ on M.

In general, the calculus of variations involves the theory of critical points of a real-valued function L on E, with $L(\phi)$ obtained by integrating a function of the derivatives of ϕ over N. The given data for a problem in *canonical form* is an r-differential form θ ($r = \dim N$) and

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