COMBINATORY RECURSIVE OBJECTS OF ALL FINITE TYPES

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In a recent paper [1] Grzegorczyk has shown that several concepts of recursive functional of finite type can be represented in a system $\mathfrak{R}$ which is closely related to combinatory logic. The exact nature of that relationship is not pointed out there, and the author appears to be unaware of it. The purpose of this note is to record some facts about this relationship; in so doing I shall dispose of an "open problem" which the author proposes in his footnote 3, p. 74. The topic is a part of an investigation now under way in preparation for the second volume of [2].[1] The notation used is that of [2]. Thus $\mathbf{F}\alpha\beta$ is Grzegorczyk's $(\alpha\beta)$; XY is his (X, Y); and $\mathbf{W}$ is his $\mathbf{D}$.

In combinatory logic, as developed previous to [2], the natural numbers were represented, following Church [3], by certain combinators $\mathbf{Z}_n$ called iterators. As stated in [2, pp. 174 ff.], a considerable arithmetic can be developed on this basis. This development is not contained in [2] for the reasons stated there on p. 7; some parts of it are, however, expounded in §§5–6 of [4] and Chapter IV of [5]. The basic ideas are due primarily to Kleene, Rosser, and Bernays. On this basis there are given in [4] two definitions of a combinator $\mathbf{R}$, called the primitive recursion combinator, such that, for any obs a, b,

$$\mathbf{R}ab\mathbf{Z}_0 = a, \qquad \mathbf{R}ab\mathbf{Z}_{n+1} = b\mathbf{Z}_n(\mathbf{R}ab\mathbf{Z}_n). \tag{1}$$

Furthermore, one can show, by natural induction on n, that for any α

$$\vdash \mathbf{F}(\mathbf{F}\alpha\alpha)(\mathbf{F}\alpha\alpha)\mathbf{Z}_n. \tag{2}$$

This motivates the assumption (here $\mathbf{N}$ is the category of natural numbers)

$$\vdash \mathbf{F}\mathbf{N}(\mathbf{F}(\mathbf{F}\alpha\alpha)(\mathbf{F}\alpha\alpha))\mathbf{I}. \tag{3}$$

Furthermore, one can show that for any α

$$\vdash \mathbf{F}_2\alpha(\mathbf{F}_2\mathbf{N}\alpha\alpha)(\mathbf{F}\mathbf{N}\alpha)\mathbf{R}. \tag{4}$$

Although the basic theory of functionality was not in existence at

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