AN EXTENSION OF THE MARCINKIEWICZ INTERPOLATION THEOREM TO LORENTZ SPACES[1]

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The purpose of this paper is three-fold. First, the theorem of Marcinkiewicz on interpolation of operators (see [9, pp. 111–116]) is generalized to Lorentz spaces [5]. Second, this result is shown to be a rather easy consequence of a celebrated inequality of Hardy [2, pp. 245–246]:

THEOREM (HARDY). *If* $q \geqq 1$, $r > 0$ *and* $f \geqq 0$, *then*

$$(1)\quad \begin{aligned} \left(\int_0^\infty \left(\int_0^t f(y)\,dy\right)^q t^{-r-1}\,dt\right)^{1/q} &\leqq \frac{q}{r}\left(\int_0^\infty (yf(y))^q y^{-r-1}\,dy\right)^{1/q};\\ \left(\int_0^\infty \left(\int_t^\infty f(y)\,dy\right)^q t^{r-1}\,dt\right)^{1/q} &\leqq \frac{q}{r}\left(\int_0^\infty (yf(y))^q y^{r-1}\,dy\right)^{1/q}. \end{aligned}$$

Third, previously open questions concerning the Marcinkiewicz Theorem are settled by showing our result is best possible.

Consider complex-valued, measurable functions f defined on a measure space (M, m). The distribution function of f is defined by

$$(2)\qquad \lambda(y) = \lambda_f(y) = m\{x \in M: |f(x)| > y\}, \qquad y > 0.$$

$\lambda(y)$ is nonincreasing and continuous from the right. The nonincreasing rearrangement of f onto $(0, \infty)$ is then defined by

$$(3)\qquad f^*(t) = \inf\{y > 0: \lambda_f(y) \leqq t\}, \qquad t > 0.$$

$f^*(t)$ is also continuous from the right and has the same distribution function as f. The Lorentz spaces $L(p, q)$ are defined to be the collection of all f such that $\|f\|^*_{pq} < \infty$, where

$$(4)\quad \|f\|^*_{pq} = \begin{cases} \left(\int_0^\infty (t^{1/p}f^*(t))^q \frac{dt}{t}\right)^{1/q}, & 0 < p < \infty,\ 0 < q < \infty,\\ \sup_{t>0} t^{1/p}f^*(t), & 0 < p \leqq \infty,\ q = \infty. \end{cases}$$

It is not hard to show that

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