A NOTE ON ISOMORPHISMS OF C^*-ALGEBRAS[1]

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1. **Introduction.** Let $\mathfrak{H}_i$, $i=1, 2$ be two Hilbert spaces of the same Hilbert dimension, $\mathfrak{L}(\mathfrak{H}_i)$, the algebra of all bounded linear operators on $\mathfrak{H}_i$. If S is any invertible, bounded linear mapping of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$, the mapping $A \rightarrow SAS^{-1}$ is an algebraic isomorphism (called "spatial") of $\mathfrak{L}(\mathfrak{H}_1)$ onto $\mathfrak{L}(\mathfrak{H}_2)$ which is a *-isomorphism (adjoint-preserving) if and only if S is unitary. This isomorphism ψ—or its restriction to a norm-closed *-subalgebra $\mathfrak{A}$ of $\mathfrak{L}(\mathfrak{H}_1)$ such that $\mathfrak{B}=\psi(\mathfrak{A})$ is also a norm-closed *-algebra—affords the most accessible illustration of an isomorphism of C^*-algebras which is not a *-isomorphism. Of course, the $\mathfrak{L}(\mathfrak{H}_i)$ are *-isomorphic, under some other maps—but what of $\mathfrak{A}$ and $\mathfrak{B}$? Even for W^*-algebras, the question has remained open: if $\mathfrak{A}$ and $\mathfrak{B}$ are algebraically isomorphic, are they necessarily *-isomorphic? See, e.g. [7, p. 1.53, Problem (i)].

In this note, the above question is answered affirmatively for the more inclusive class of C^*-algebras [Theorem 3].

Theorem 2 gives the structure of isomorphisms of C^*-algebras, showing that each is, in a certain canonical sense, spatial in nature. The Invariance Theorem 1 stems from the theory of analytic functions in Banach algebras, and is employed with Theorem 2 to prove Theorem 3.

The proofs will be sketched. Full details will appear elsewhere.

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2. **Preliminaries: Representation theory.** By C^*-algebra we mean an abstract complex Banach *-algebra $\mathfrak{A}$ with $\|A^*A\| = \|A^*\| \, \|A\|$ for all $A \in \mathfrak{A}$ (B^*-algebra). A representation (*-representation) of $\mathfrak{A}$ on the Hilbert space $\mathfrak{H}$ is a homomorphism (*-homomorphism) of $\mathfrak{A}$ into $\mathfrak{L}(\mathfrak{H})$, the algebra of all bounded operators on $\mathfrak{H}$. A *-representation is of norm at most 1, and its image is norm-closed. A *-representation ϕ on $\mathfrak{H}$ is *cyclic* if there exists a vector x in $\mathfrak{H}$ (cyclic vector) such that the closure $[\phi(\mathfrak{A})x]$ of $\{\phi(A)x \mid A \in \mathfrak{A}\}$ is $\mathfrak{H}$. It is *irreducible* if every

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