THE CLOSURE OF THE NUMERICAL RANGE CONTAINS THE SPECTRUM[1]

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The purpose of this research is to prove that the well-known theorem in the theory of linear operators in Hilbert space [1, p. 147] indicated in the title holds for nonlinear operators and to a certain extent for noncontinuous ones, and to provide a constructive method for solving the equations involved.

DEFINITION 1. The numerical range of a mapping $T: \mathcal{H} \to \mathcal{H}$ of a complex Hilbert space into itself with domain $\mathfrak{D}(T)$ is the set of complex numbers

$$\mathfrak{N}(T) = \left\{ \frac{(Tx_1 - Tx_2, x_1 - x_2)}{\|x_1 - x_2\|^2}, \quad x_1 \neq x_2, \quad x_1, x_2 \in \mathfrak{D}(T) \right\}.$$

In the case of linear mappings, we recall, this is a convex set, which, if the mapping is maximal normal, has a closure coinciding with the convex hull of the spectrum [1, pp. 131, 327].

We shall let $\|T\|$ denote the Lipschitz norm of T, namely,

$$\|T\| = \sup_{x_1 \neq x_2} \frac{\|Tx_1 - Tx_2\|}{\|x_1 - x_2\|}.$$

We shall also use the weaker norm—called the cross-Lipschitz norm —that results from replacing in the above definition the increment $Tx_1 - Tx_2$ by its component orthogonal to $x_1 - x_2$. In general, we define the ν-cross-Hölder norm $(0 \leqq \nu \leqq 1)$ as the quantity

$$\|T\|_\nu^\perp = \sup_{x_1 \neq x_2} \left\{ \left[\|Tx_1 - Tx_2\|^2 - \frac{|(Tx_1 - Tx_2, x_1 - x_2)|^2}{\|x_1 - x_2\|^2} \right]^{1/2} \Big/ \|x_1 - x_2\|^\nu \right\}.$$

If $\|T\|_\nu^\perp < \infty$ we say that T satisfies a cross-Hölder condition of exponent ν. The cross-Lipschitz norm corresponds to $\nu = 1$ and shall simply be denoted $\|T\|^\perp$; for finite-dimensional normal linear mappings it measures the size of the spectrum. In these definitions we have implicitly assumed that the variables range over the whole do-

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