ON NORMAL METRICS, AND A THEOREM OF COHN-VOSSEN

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1. Let $\mathcal{G}$ denote the domain consisting of the complex z-plane with $n+1$ points $p_1, \cdots, p_n, p_0=\infty$ deleted. Let $\mu(E)$ be a measure over $\mathcal{G}$, of finite total variation. Let

$$u(x, y) = \int_{\mathcal{G}} \log \left| 1 - \frac{z}{\zeta} \right| d\mu_\zeta + h(z),$$

where $h(z)$ is harmonic in $\mathcal{G}$. The conformal metric $e^{u(z)}|dz|$ will be said to be *normal* in $\mathcal{G}$, provided $h(z)$ has the form

$$h(z) = \sum_1^n \beta_j \log |z - p_j| + \text{const.}$$

The metric is said to be *complete* if any path tending to one of the $\{p_j\}$ has infinite length. The *curvatura integra* is $C = -2\pi \int_{\mathcal{G}} d\mu_\zeta$, and the *flux* at p_j is defined by

$$\Phi_j = \lim_{\gamma_j \to p_j} \frac{1}{2\pi} \oint_{\gamma_j} \frac{\partial u}{\partial n} |dz|,$$

for curves γ_j enclosing p_j.

Let Γ_j, γ_j be concentric circumferences surrounding p_j, and let $\mathfrak{A}_j(\Gamma, \gamma)$ be the area of the enclosed annulus in the given metric. Let $\mathfrak{L}_j(\gamma)$ be the length of γ_j.

THEOREM. *If the metric is complete, then* $\Phi_0 \geqq -1, \Phi_j \geqq 1, j>0$.[1] *The quantities*

$$\nu_j = \lim_{\gamma_j \to p_j} \frac{\mathfrak{L}_j(\gamma)}{4\pi \mathfrak{A}_j(\Gamma, \gamma)}$$

exist and are finite for each j, *and* $\nu_0 = \Phi_0 + 1$, $\nu_j = \Phi_j - 1$, $j>0$. *There holds*

$$C = 2\pi\left(\chi - \sum_0^n \nu_j\right),$$

where $\chi = 1 - n$ *is the Euler characteristic of* $\mathcal{G}$.

[1] This assertion follows alternatively from the work of Huber [2]. The demonstrations differ.