# RESEARCH PROBLEMS

10. R. M. Redheffer: *Operators on Hilbert space.*

Let $u$, $r$, $s$, $w$, $z$ denote closed linear operators defined on a Hilbert space $H$, with $r\neq 0$, $s\neq 0$ and $\|w\| \leqq 1$. Define operators

$$f(z) = u + rz(1 - wz)^{-1}s, \qquad S_\lambda = \begin{pmatrix} r\lambda & u \\ w & \lambda^{-1}s \end{pmatrix}$$

on $H$ and $H \times H$, respectively, $\lambda$ being a positive scalar. As norm $\|u\|$ we take sup $|uv|$ for $v \in H$, $|v| = 1$, and similarly in other cases, such as $\|S_\lambda\|$. Lengths on $H \times H$ are related to those on $H$ by

$$|(v_1, v_2)|^2 = |v_1|^2 + |v_2|^2, \qquad v_i \in H.$$

*Problem* A. Give a simple proof of the following: If $\|f(z)\| \leqq 1$ for all $\|z\| \leqq 1$ such that $(1 - wz)^{-1}$ exists, then $\|S_\lambda\| \leqq 1$ for some $\lambda$.

*Problem* B. Give a simple proof of this: If sup $\|f(z)\| < 1$ for $\|z\| \leqq 1$, then $f(z)$ has a fixed point in $\|z\| < 1$.

*Problem* C. What happens in Problem B if we only have $\|f(z)\| \leqq 1$ for $\|z\| \leqq 1$?

*Problem* D. Let $U$ denote the class of unitary operators, and $N$ the class with norm $\leqq 1$. Study the class of functions $h(z)$ that satisfy a "maximum principle" in the following sharp form:

$$\sup_{z\in N} \|h(z)\| = \sup_{z\in U} \|h(z)\|.$$

In Problems A and B the emphasis is on the word "simple." Both results have been established, but the only known proof is harder than the depth of the problems seems to warrant. I expect a simple proof because: the converse of Problem A is easy; both problems are easy when the unit ball is compact, e.g., matrices; the two problems are easily proved equivalent to each other; the appropriate form of Problem A when "$\|f(z)\| \leqq 1$ for $\|z\| \leqq 1$" is replaced by "$f(z)$ unitary for $z$ unitary" is easy; and the fact that $f(z)$ can be written $(a + bz)(c + dz)^{-1}$ suggests connections with many well-known theories.

In Problem D the theory developed should include the known fact that $f(z)$ has the stated property when $\|w\| < 1$. (Received July 7, 1964.)