THE STABLE STRUCTURE OF QUITE GENERAL LINEAR GROUPS

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1. **Introduction.** Dieudonné [5] determined the normal subgroups of $\mathrm{GL}(n, A)$ for an (even noncommutative) field, A, and Klingenberg recently showed [7; 8] that Dieudonné's result, suitably formulated, survives without surprises for A any local ring. The results described here constitute the beginnings of a global theory. The information they yield on $\mathrm{SL}(n, \boldsymbol{Z})$, combined with a rather formidable cohomological calculation, is the basis of the proof in [3] that every subgroup of finite index in $\mathrm{SL}(n, \boldsymbol{Z})$, $n \geqq 3$, contains a congruence subgroup.

This material, the details of which will appear in [1], is based on the algebraic K-theory described in [4]. The topological intuition thereby afforded intervenes via the space, X, of maximal ideals of a commutative ring, A. Thus, our results on $\mathrm{GL}(n, A)$ are effective only if n is sufficiently large compared with dim X, i.e. only if n is in the stable range. If A is semi-local this is no restriction, X being then finite. If $A = \boldsymbol{Z}$ then dim $X = 1$. For general A we must let n go to infinity (stabilize) (§2). While the Dieudonné-Klingenberg theorem may fail even then, its failure is measured by certain abelian groups, $K^1(A, \mathfrak{q})$, one for each ideal $\mathfrak{q}$. When $\mathfrak{q} = A$ we write $K^1(A) = K^1(A, A)$. When dim $X = 0$ they reduce to something essentially trivial, and we recover Dieudonné-Klingenberg.

In a joint paper with A. Heller and R. Swan [2] the homomorphisms $K^1(A) \to K^1(A[t])$ and $K^1(A) \to K^1(A[t, t^{-1}])$, t an indeterminate, are analyzed (see §5). Concerning the latter Atiyah has pointed out that our result is an analogue of Bott periodicity for the unitary group (see §6).

Finally, various of these results yield information (§7) on J. H. C. Whitehead's groups of simple homotopy types [9], results which extend some earlier work of G. Higman [6].

2. **Stable structure theorem.** For a ring A, $\mathrm{E}(n, A)$ denotes the subgroup of $\mathrm{GL}(n, A)$ generated by all elementary matrices, i.e. those differing from the identity in a single, off diagonal, coordinate. If $\mathfrak{q}$ is an ideal we write

$$\mathrm{GL}(n, A, \mathfrak{q}) = \ker(\mathrm{GL}(n, A) \to \mathrm{GL}(n, A/\mathfrak{q})),$$

and $\mathrm{E}(n, A, \mathfrak{q})$ denotes the *normal* subgroup of $\mathrm{E}(n, A)$ generated by the elementary matrices in $\mathrm{GL}(n, A, \mathfrak{q})$.