THE EQUIVALENCE, FOR VARIETIES OF SEMIGROUPS, OF TWO PROPERTIES CONCERNING CONGRUENCE RELATIONS

BY E. J. TULLY, JR.[1]

An equivalence relation σ on a semigroup S is called a congruence if $a\,\sigma\,b$ and $c\,\sigma\,d$ imply $(ac)\sigma\,(bd)$, for all $a, b, c, d \in S$. There is a fairly obvious correspondence between congruences on S and homomorphic images of S. For notational convenience we shall denote by S' the semigroup S with an identity element (if one is not already present) adjoined.

If $a \in S$, there is at least one congruence (viz., the identity relation, having a single element in each of its equivalence classes) for which $\{a\}$ is an equivalence class. Teissier [5] has essentially shown that this congruence is the only one having $\{a\}$ as a class if and only if:

(1) For each $b, c \in S$ with $b \neq c$, there exist $x, y \in S'$ such that exactly one of the pair xby, xcy equals a.

We shall call S *disjunctive* if (1) holds for all $a \in S$. Thus we may say that a disjunctive semigroup is characterized by the property that the identity relation is uniquely determined by each of its classes.

Now suppose σ and ρ are congruences on S. We define a relation $\sigma \circ \rho$, called the product of σ and ρ, by: $a\,(\sigma \circ \rho)\,b$ if and only if $a\,\sigma\,x$ and $x\,\rho\,b$ for some $x \in S$. The assumption that every pair σ, ρ of congruences on S is *permutable* (in the sense that $\sigma \circ \rho = \rho \circ \sigma$) has a number of interesting consequences, e.g., an analogue of the Jordan-Hölder Theorem, in which one speaks of chains of congruences in place of chains of subgroups. See Birkhoff [1], Chapter VI (especially Theorem 5, page 87), where further references are given.

It is easy to see that the two conditions which we have considered (viz., disjunctivity and congruence-permutability) are not equivalent. For let $S = \{1, 2, \cdots, n\}$, where $n \geq 3$, with the semigroup operation given by: $x \circ y = x + y$ if $x + y \leq n$, $x \circ y = n$ if $x + y > n$. Then S is congruence-permutable, but not disjunctive.

A family V of semigroups is called a *variety* of semigroups if V contains all subsemigroups, all homomorphic images, and all direct prod-