SOME CURIOUS INVOLUTIONS OF SPHERES

BY MORRIS W. HIRSCH AND JOHN MILNOR

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Consider an involution T of the sphere S^n without fixed points. Is the quotient manifold S^n/T necessarily isomorphic to projective n-space? This question makes sense in three different categories. One can work either with topological manifolds and maps, with piecewise linear manifolds and maps, or with differentiable manifolds and maps.

For $n \leqq 3$ the statement is known to be true (Livesay [6]). In these cases it does not matter which category one works with. On the other hand, for $n=7$, in the differentiable case, the statement is known to be false (Milnor [10]).

This note will show that, in the piecewise linear case, the statement is false for all $n \geqq 5$. Furthermore, for $n=5, 6$, we will construct a differentiable involution $T: S^n \to S^n$ so that the quotient manifold is not even piecewise linearly homeomorphic to projective space. Our proofs depend on a recent theorem of J. Cerf.

Let us start with the exotic 7-sphere M_3^7 as described by Milnor [7]. This differentiable manifold M_3^7 is defined as the total space of a certain 3-sphere bundle over the 4-sphere. It is known to be homeomorphic, but not diffeomorphic, to the standard 7-sphere.

Taking the antipodal map on each fibre we obtain a differentiable involution $T: M_3^7 \to M_3^7$ without fixed points. (The quotient manifold M_3^7/T can be considered as the total space of a corresponding projective 3-space bundle over S^4.) The following lemma was pointed out to us, in part, by P. Conner and D. Montgomery.

LEMMA 1. *There exists a differentiably imbedded 6-sphere,* $S_0^6 \subset M_3^7$, *which is invariant under the action of* T, *and a differentiably imbedded* $S_0^5 \subset S_0^6$ *which is also invariant.*

Thus in this way one constructs a differentiable involution of the standard sphere in dimensions 5, 6.

The proof will depend on the explicit description of M_3^7 (or more generally of M_k^7) which was given in [7]. Take two copies of $R^4 \times S^3$ and identify the subsets $(R^4-(0)) \times S^3$ under the diffeomorphism

$$(u, v) \to (u', v') = (u/\|u\|^2, u^h v u^j/\|u\|),$$

using quaternion multiplication, where $h+j=1$, $h-j=k$. The involution T changes the sign of v and v'. Let S_0^6 be the set of all points of M_k^7 such that $\Re(v') = \Re(uv) = 0$, where $\Re(uv) = \Re(vu)$ denotes the real