ON THE LIFTING PROPERTY. III[1]

BY A. IONESCU TULCEA AND C. IONESCU TULCEA

Communicated by Walter Rudin, September 9, 1963

1. Let Z be a locally compact space and $\mu \neq 0$ a positive Radon measure on Z. Let $M_R^\infty(Z, \mu)$ be the Banach algebra of all bounded real-valued μ-measurable functions defined on Z, endowed with the norm $f \rightarrow \|f\|_\infty = \sup_{z \in Z} |f(z)|$. Let $C_R^\infty(Z)$ be the subalgebra of $M_R^\infty(Z, \mu)$ consisting of all bounded continuous functions on Z and $\mathcal{K}(Z)$ the subalgebra of all $f \in C_R^\infty(Z)$ having compact support. For two functions f and g defined on Z we shall write $f \equiv g$ whenever f and g coincide locally almost everywhere.

Let now $T: f \rightarrow T_f$ be a mapping of $M_R^\infty(Z, \mu)$ into $M_R^\infty(Z, \mu)$. Properties of T such as those listed below will be considered in what follows:

(I) $T_f \equiv f$;

(II) $f \equiv g$ implies $T_f = T_g$;

(III) $T_1 = 1$;

(IV) $f \geqq 0$ implies $T_f \geqq 0$;

(V) $T_{\alpha f + \beta g} = \alpha T_f + \beta T_g$;

(VI) $T_{fg} = T_f T_g$;

(VII) $T_f = f$ if $f \in C_R^\infty(Z)$.

A mapping $T: f \rightarrow T_f$ of $M_R^\infty(Z, \mu)$ into $M_R^\infty(Z, \mu)$ satisfying (I)–(V) will be called a *linear lifting* of $M_R^\infty(Z, \mu)$; if the condition (VI) is also verified the mapping will be called a *lifting* of $M_R^\infty(Z, \mu)$. A *strong linear lifting* [*strong lifting*] of $M_R^\infty(Z, \mu)$ is a linear lifting [lifting] which verifies also (VII).

If $T: f \rightarrow T_f$ is a lifting of $M_R^\infty(Z, \mu)$ and A is a μ-measurable set then we shall denote by $\rho_T(A)$ the set defined by the equation[2] $T_{\phi_A} = \phi_{\rho_T(A)}$. For each $z \in Z$ denote by $\mathcal{V}_T(z)$ the set of all parts $\rho_T(V)$ where V belongs to[3] $\mathcal{V}(z)$ and is μ-measurable. If τ_T is the set of all parts $\rho_T(A) - N$ where A is μ-measurable and N is locally μ-negligible then τ_T is a topology on Z (this result is essentially due to J. Oxtoby and has been given by him in a lecture at Yale in the fall of 1960).

THEOREM 1. *Let $T: f \rightarrow T_f$ be a lifting of $M_R^\infty(Z, \mu)$. Then the following assertions are equivalent*: (1.1) *T is a strong lifting*; (1.2) *There is*

[1] Research supported by the U. S. Army Research Office (Durham) under contract DA-ARO(D)-31-124-G218.

[2] For $X \subset Z$, ϕ_X denotes the characteristic function of X.

[3] For each $z \in Z$, $\mathcal{V}(z)$ denotes the set of all neighborhoods of z.