CONVERGENCE OF SEQUENCES OF CONVEX SETS, CONES AND FUNCTIONS

BY R. A. WIJSMAN

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1. **Introduction and summary.** We shall consider sequences of subsets of Euclidean m-space, especially convex sets that are not necessarily bounded. A notion of convergence of a sequence of sets to a set is introduced, having the property that convergence of convex sets implies convergence of the corresponding polars, and, under a mild condition, of the corresponding projecting cones. It is, however, not true that convergence of convex sets is equivalent to pointwise convergence of the corresponding support functions. Only after introduction of a new type of convergence of a sequence of functions to a function (termed *infimal convergence*) is the desired equivalence achieved. The relation between a sequence of closed convex functions and the sequence of their conjugates is studied. It turns out that either sequence converges infimally if and only if the other does. Finally, infimal convergence of closed convex functions implies convergence of their level sets, under a mild condition. Most of the theorems are valid in more general topological spaces, and sequences may be replaced by nets throughout. Proofs, lemmas and additional theorems will appear elsewhere.

2. **Definitions and notation.** Let O denote the origin. If $x \neq O$, a ray from O through x is denoted (x). A *cone* is a union of rays. In the space of rays a metric can be introduced by identifying each ray with its intersection with the unit $(m-1)$-sphere, and taking (for instance) the chord distance topology on the $(m-1)$-sphere. This defines *open cone*, etc. The *projecting cone* of a set X is $P(X) = \{(x) : x \in X\}$. The *asymptotic cone* of X is $A(X) = \{(x) : (x) = \lim (x_n), x_n \in X, |x_n| \to \infty\}$, where $| \ |$ denotes Euclidean norm. The *distance function* $d(X)$ of X is defined by $d(X, x) = \inf\{|x-y| : y \in X\}$. The *support function* $h(X)$ of X is defined by $h(X, \xi) = \sup_{x \in X} \xi \cdot x$, where $\cdot$ denotes inner product. D_R is the m-disk of radius R. Closure of a set X is denoted by $\overline{X}$ or by $\mathrm{Cl}[X]$.

3. **Convergence of sets and projecting cones.** If X_n, X are sets, we shall define $X_n \to X$ if $d(X_n) \to d(X)$ pointwise. For closed limit sets this corresponds to a definition given by Frolík [3] for general topological spaces. If the sets are closed, it is possible to introduce a