THE DIMENSION OF THE SUPPORT OF A RANDOM DISTRIBUTION FUNCTION

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In their paper *Random distribution functions* (Bull. Amer. Math. Soc. **69** (1963), 548–551) L. E. Dubins and D. A. Freedman defined a random distribution function F associated with a probability measure μ on the unit square S whose values are distribution functions on $[0, 1]$. To choose a value F_ω of F they proceed as follows: Points $P(n, j)$ of S are defined inductively for all n and $j=0, \cdots, 2^n$ by setting $P(0, 0)=(0, 0)$, $P(0, 1)=(1, 1)$, $P(n+1, 2j)=P(n, j)$ and $P(n+1, 2j+1)$ equal to the image under the unique affine transformation carrying S onto the rectangle $R(P(n, j), P(n, j+1))$ formed by the vertical and horizontal lines through $P(n, j)$ and $P(n, j+1)$ of a point $P^*(n+1, 2j+1)=(x^*(n, 2j+1), y^*(n, 2j+1))$ chosen according to the distribution μ independently of the previous choices. They showed that $\bigcap_{n=1}^{\infty} \bigcup_{j=0}^{2^n} R(P(n, j), P(n, j+1))$ is the graph of a continuous monotone function $F_\omega(x)$ increasing from 0 to 1 on $[0, 1]$, that is, a distribution function defining a measure $\tilde{F}_\omega(E) = \int_E dF_\omega(x)$ on measurable $E \subset [0, 1]$. The inverse of $F_\omega(x)$ is also a continuous everywhere increasing function which we call $G_\omega(y)$ with corresponding measure $\tilde{G}_\omega(E)$. Let

$$I(n, j) = [x(n, j-1), x(n, j)],$$
$$J(n, j) = [y(n, j-1), y(n, j)]$$

and

$$I(n, x) = I(n, j), J(n, x) = J(n, j) \text{ for that } j \text{ for which } x \in I(n, j).$$

$I(n, y)$ and $J(n, y)$ are defined similarly. Let $I^*(n, 2j+\epsilon) = [0, x^*(n, 2j+1)]$ or $[x^*(n, 2j+1), 1]$ and $J^*(n, 2j+\epsilon) = [0, y^*(n, 2j+1)]$ or $[y^*(n, 2j+1), 1]$ according as ϵ equals 0 or 1. We shall write $|I|$ for the length of the interval I, and $h(a, b)$ for the function on S given by $h(a, b)=a \log b+(1-a) \log_2 (1-b)$. All logarithms are taken to the base 2. For any function $k(x, y)$ on S we set

$$E_\mu(k(x, y)) = \int_0^1 \int_0^1 k(x, y) d\mu(x, y)$$

and

$$\sigma_\mu^2(k(x, y)) = E_\mu([k(x, y) - E_\mu(k(x, y))]^2).$$