WEIGHTED TRIGONOMETRICAL APPROXIMATION ON R^1 WITH APPLICATION TO THE GERM FIELD OF A STATIONARY GAUSSIAN PROCESS

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Given an even, nonnegative, Lebesgue measurable weight $\Delta=\Delta(a)$ $(a\in R^1)$ with $\int\Delta<\infty$, let Z be the (real) Hilbert space of Lebesgue measurable functions f with $f^*(-a)=f(a)$ and $\|f\| =\sqrt{\int|f|^2\Delta}<\infty$, subject to the usual identifications, let Z^{cd} be the span (in Z) of e^{iat} $(c\leqq t\leqq d)$, and introduce the following subspaces of Z:

(a) $Z^-=Z^{-\infty 0}$,

(b) $Z^+=Z^{0\infty}$,

(c) $Z^{+/-}=$ the projection of Z^+ upon Z^-,

(d) $Z^\bullet=$ the class of entire functions $f=f(\gamma)$ $(\gamma=a+ib)$ of minimal exponential type which, restricted to the line $b=0$, belong to Z,

(e) $Z^{0+}=\bigcap_{\delta>0} Z^{0\delta}$,

(f) $Z_\bullet=$ the span of (real) polynomials of ia belonging to Z,

(g) $Z^{-\infty}=\bigcap_{t<0} Z^{-\infty t}$.

Δ is a Hardy weight if

$$\int\frac{lg^-\Delta}{1+a^2}>-\infty;$$

such a Hardy weight is expressible as $|h|^2$, h being an (outer) function belonging to the Hardy class of functions $f(\gamma)$ $(\gamma=a+ib)(\gamma^*=a-ib)$ regular in the half plane $(b>0)$ with $f^*(-a)=f(a)$ and $\int|f(a+ib)|^2da$ bounded $(b>0)$. $Z\neq Z^-$ or $Z=Z^-=Z^{-\infty}$ according as Δ is Hardy or not, a fact that goes back to Szegö.

Given a Hardy weight, it can be proved that

$$Z^-\supset Z^{+/-}\supset Z^-\cap Z^+\supset Z^\bullet\supset Z^{0+}\supset Z_\bullet,$$

and the problem is to decide if some or all of the above subspaces coincide, special attention being paid to $Z^{+/-}$ and Z^{0+} for probabilistic reasons explained below. $Z^{0+}=Z^\bullet$ for the general Hardy weight, but the other inclusions can be strict; for instance, $Z^-\neq Z^{+/-}$ if and only if $\mathfrak{j}=h/h^*$, restricted to the line $b=0$, agrees with the ratio of two inner functions, while $Z^{+/-}=Z^\bullet$ $(=Z^{0+})$ if and only if the reciprocal h^{-1} of the outer Hardy function h figuring in $\Delta=|h|^2$ is an entire function of minimal exponential type. $Z^\bullet\neq Z_\bullet$ is possible even for such

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