AN ALGEBRAIC MODEL OF TRANSITIVE DIFFERENTIAL GEOMETRY

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One of the basic assumptions that is frequently made in the study of geometry (or physics) is that of homogeneity. It is assumed that any point can be moved into any other by a transformation preserving the underlying geometrical structure (e.g., by a Euclidean motion in Euclidean geometry, an inhomogeneous Lorentz transformation in special relativity, etc.). The problem we shall treat in this paper is that of describing the possible differential geometries which are homogeneous or *transitive* in the above sense. We prefer the word "transitive" as "homogeneous" already has a technical meaning. It should be noted at the outset that we will only discuss the local problem, regarding two geometries as the same if they are locally the same. The problem of describing the global possibilities for a fixed geometrical structure is a problem of an entirely different order and is usually extremely interesting and difficult (the problem of moduli for Riemann surfaces, the Clifford-Klein Raum problem, etc.). Actually, we won't even handle the local problem which involves various technical difficulties such as solving partial differential equations, worrying about domains of definition, etc. What we shall do is construct an algebraic model which will illustrate all the crucial geometric notions.

Before giving a precise statement to the problem let us examine the situation in two dimensions. In order to list all the possible transitive geometries we must first list the various types of differential geometry (projective, conformal, Riemannian, etc.) and then list the possibilities for each type. For instance, the Korn-Lichtenstein theorem (asserting the existence of isothermal coordinates (cf. [10] or [25])) implies that all conformal differential geometries in two dimensions are locally equivalent. In other words there is only one conformal differential geometry and it is transitive.

How many different transitive Riemannian geometries are there in two dimensions? Here we are faced with a choice. The professional differential geometer will answer that there is a continuum, each parametrized by a real number—the curvature. Indeed a sphere of

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