FREDHOLM EIGENVALUES AND QUASICONFORMAL MAPPING[1]

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Let $\tilde{D}$ be a region of connectivity n in the z-plane which contains the point at infinity. Suppose that its boundary consists of n Jordan curves $C_1, C_2, \cdots, C_n$, each of which is given parametrically in terms of its arc length by functions having continuous second derivatives which satisfy a Hölder condition of order α $(0 < \alpha \leqq 1)$. Each C_j $(j=1, \cdots, n)$ forms the boundary of a simply connected bounded region D_j, and we shall write $D=D_1\cup \cdots \cup D_n$ and $C=C_1\cup \cdots \cup C_n$.

Let λ denote the smallest eigenvalue satisfying $\lambda>1$ of the integral equation

$$f(z) = \frac{\lambda}{\pi}\int_C f(t)\,\frac{\partial}{\partial n_t}\log\frac{1}{|z-t|}\,ds_t, \qquad t\in C,$$

where s_t denotes the arc length parameter on C oriented positively with respect to $\tilde{D}$ and $\partial/\partial n_t$ denotes differentiation in the direction of the normal to C pointing into $\tilde{D}$. We shall refer to λ as the Fredholm eigenvalue of C.

We next suppose that $\zeta(z)$ is a quasiconformal homeomorphism of the z-sphere onto the ζ-sphere $(\infty\to\infty)$ whose generalized derivatives are denoted by

$$p = \frac{\partial\zeta}{\partial z} = \frac{1}{2}\left(\frac{\partial\zeta}{\partial x} - i\frac{\partial\zeta}{\partial y}\right) \quad\text{and}\quad q = \frac{\partial\zeta}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial\zeta}{\partial x} + i\frac{\partial\zeta}{\partial y}\right).$$

We assume that $\zeta(z)$ is K-quasiconformal in D and M-quasiconformal in $\tilde{D}$; i.e., $\|q\|_\infty \leqq k\|p\|_\infty$ $(k<1)$ in D and $\|q\|_\infty\leqq m\|p\|_\infty$ $(m<1)$ in $\tilde{D}$, where $K=(1+k)/(1-k)$ and $M=(1+m)/(1-m)$. The image of each C_j is a curve C_j^* and the curve system $C=C_1^*\cup\cdots\cup C_n^*$ has Fredholm eigenvalue λ^*. The following inequality holds:

THEOREM.

$$\frac{\lambda^*+1}{\lambda^*-1}\leqq KM\frac{\lambda+1}{\lambda-1}. \tag{1}$$

If λ is known and the K and M for the mapping ζ are known, then

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